1 Due Tuesday, September 3

Multiple choice questions

Multiple choice scoring. Each correctly identified choice is worth 1 positive point. For every incorrect choice, you will receive a deduction of 1 point. Yes, it is possible to get a negative score on these multiple choice problems! So think carefully.

You do *not* need to give any justification for your multiple choice responses.

Remark 1.0.1. Remember, in mathematics, "true" means always true. "False" means *not* always true.

In some of the problems below, I will carefully remind you of what it means for particular statements to be true. I will remove these parenthetical clarifications in the days to come.

Preliminaries

Given two real numbers x and x',[1] we define

$$d(x, x') = |x - x'|.$$

1.1

Which of the following statements are true?

(a) $d(x, x') = d(x', x)$. That is, d is *symmetric*. (For this statement to be true, we man that for *any* choice of x and any choice of x', the equality holds.)

(b) If $d(x, x') = 0$, then $x = x'$.

(c) Let x, x', x'' be three real numbers. Then $d(x, x') + d(x', x'') \leq d(x, x'')$. (Remember, that this statement is true means that for *any* choice of three real numbers x, x', and x'', this inequality holds.)

[1] Recall that in math-talk, even though we say "two" real numbers x and x', our choices may include the case where x is the same real number as x'.

(d) $d(x, x')$ may be negative. (That is, there exists a pair of real numbers x and x' for which $d(x, x')$ is a negative number.)

(e) Let x, x', x'' be three real numbers. If $d(x, x') = d(x, x'')$ then $x' = x''$.

1.2

Which of the following statements are true?

(a) (Scaling.) For all real numbers k, x, x', we have

$$d(kx, kx') = kd(x, x').$$

(b) (Translation invariance.) For all real numbers x, x', x'', we have

$$d(x - x'', x' - x'') = d(x, x').$$

(c) For all real numbers x, x', x'', we have

$$d(x + x'', x') = d(x, x') + x''.$$

(d) (Skew-symmetry.) $d(x, x') = -d(x', x)$.

1.3

Fix an integer $n \geq 1$.[2] We let $X = \mathbb{R}^n$, so that an element $x \in X$ is the data of n real numbers, called the *coordinates* of x. We will denote the coordinates of x by

$$(x_1, \ldots, x_n).$$

So for example, $(\pi, e, 13)$ is an element of $\mathbb{R}^3$.

We define

$$d(x, x') = \sqrt{(x_1 - x_1')^2 + \ldots + (x_n - x_n')^2}.$$

Which of the following are true?

(a) For any two $x, x' \in \mathbb{R}^n$, we have $d(x, x') = d(x', x)$. That is, d is *symmetric*.

(b) For any two $x, x' \in \mathbb{R}^n$, if $d(x, x') = 0$, then $x = x'$.

(c) For any three $x, x', x'' \in \mathbb{R}^n$, we have $d(x, x') + d(x', x'') \leq d(x, x'')$.

[2]Remember, this means that n—in what follows—is any integer greater than or equal to 1.

1.4

For $n \geq 0$, we let $0 \in \mathbb{R}^{n+1}$ denote the origin; this is the element whose coordinates are all equal to zero.

We let S^n denote the set of all points x' such that $d(0, x') = 1$.

Which of the following are true?

(a) ($n = 0$.) S^0 consists of exactly two points.

(b) ($n = 1$.) S^1 is a circle.

(c) ($n = 2$.) S^2 is a sphere.

(d) ($n = 3$.) S^3 is a cube.

Proofs

1.5 (10 points)

Fix an infinite sequence of real numbers $x_1, x_2, \ldots,$.

Definition 1.5.1. We say that the sequence *converges*, or is *convergent*, if there exists a real number x such that the following holds:

For every $\epsilon > 0$, there exists an integer N such that

$$i > N \implies |x_i - x| < \epsilon.$$

Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and if a sequence $x_1, x_2, \ldots$ converges, then the sequence $f(x_1), f(x_2), \ldots$ also converges.

1.6 Extra Credit (5 points)

Investigate the converse to the previous problem. That is, suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying the following property: For any convergent sequence $x_1, x_2, \ldots$, the sequence $f(x_1), f(x_2), \ldots$ converges.

Is it true that f must be continuous? Give a complete proof or counterexample.

1.7 Extra Credit (5 points)

In this problem, you will prove that any pair of closed, bounded intervals of non-zero length are homeomorphic.

So fix real numbers $a < b$ and $c < d$. Prove that there exists a continuous function

$$f : \mathbb{R} \to \mathbb{R}$$

satisfying the following properties:

1. f takes the interval $[a, b]$ to $[c, d]$ bijectively.
2. Moreover, the inverse

   $$g : [c, d] \to [a, b]$$

   is continuous.

1.8 Extra Credit (5 points)

Does there exists a continuous bijection f from $[a, b]$ to $[c, d]$ such that the inverse function from $[c, d]$ to $[a, b]$ is *not* continuous?