2 Due Tuesday, September 10

As before, here is how I score multiple choice problems: A correct choice gets +1 point. An incorrect choice results in a deduction of 1 point. It is possible to get a negative score on the multiple choice.

It will be useful to know the following:

Definition 2.0.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A function

$$f : X \to Y$$

is called *continuous* if for all $x \in X$ and for all $\epsilon > 0$, there exists $\delta > 0$ so that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon.$$

Definition 2.0.2. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called an *isometry* if

1. f is a bijection, and
2. for all $x, x' \in X$, we have

$$d_X(x, x') = d_Y(f(x), f(x')).$$

2.1

Which of the following choices of (X, d) is a metric space?

(a) $X = \mathbb{R}$, and $d(x, x') = x$.

(b) $X = \mathbb{R}$, and $d(x, x') = x - x'$.

(c) $X = \mathbb{R}$, and $d(x, x') = |x - x'|$.

(d) $X = \mathbb{R}$, and $d(x, x') = \sqrt{(x' - x)^2}$.

(e) $X = \mathbb{R}^3$ and $d(x, x') = |x_1 - x_1'| + |x_2 - x_2'| + |x_3 - x_3'|$.

(f) $X = \mathbb{R}^3$ and $d(x, x') = |x_1 - x_1'| + |x_2 - x_2'|$.

(g) X is any set, and $d(x, x') = 0$.

(h) X is any set, and

$$d(x, x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x'. \end{cases}$$

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2.2

Let (X, d_X) and (Y, d_Y) be metric spaces. Which of the following statements is true?

(a) An isometry $f : X \to Y$ is a continuous function.

(b) If $f : X \to Y$ is an isometry, then its inverse function is also an isometry.

(c) If $f : X \to Y$ is continuous, then f is a bijection.

(d) If $f : X \to Y$ is a bijection, then f is an isometry.

(e) Let (Z, d_Z) be another metric space. If $f : X \to Y$ is an isometry and $g : Y \to Z$ is an isometry, then the composition $g \circ f : X \to Z$ is an isometry.

(f) Let (Z, d_Z) be another metric space. If $f : X \to Y$ is continuous and $g : Y \to Z$ is continuous, then the composition $g \circ f : X \to Z$ is continuous.

2.3 (10 points)

Let (X, d_X) and (Y, d_Y) be metric spaces. We endow the product $X \times Y$ with the following metric:

$$d((x, y), (x', y')) = d(x, x') + d(y, y').$$

Prove the following:

(a) The projection map

$$X \times Y \to X, \qquad (x, y) \mapsto x$$

is continuous.

(b) Let Z be a metric space and fix two continuous functions $f : Z \to X$ and $g : Z \to Y$. Then the function

$$f \times g : Z \to X \times Y, \qquad (f \times g)(z) = (f(z), g(z))$$

is continuous.

(c) Let $C^0(Z, X)$ denote the set of continuous functions from Z to X. Prove that the product set
$$C^0(Z, X) \times C^0(Z, Y)$$
is in bijection with the set
$$C^0(Z, X \times Y).$$

2.4 Extra credit (10 points)

Write out a complete explanation for why each of the multiple choice options from Homework One are either true, or false.

These explanations must be succinct and correct for full credit.

2.5 Extra credit (5 points)

Let $\mathbb{R}^\infty$ denote the set of all infinite sequences
$$x = (x_1, x_2, \ldots)$$
for which all but finitely many x_i equal 0. You do not need to prove this, but the function
$$d(x, x') = \sum_{i=1}^{\infty} |x'_i - x_i|$$
renders $\mathbb{R}^\infty$ a metric space.

Show that for any metric space Z, a function
$$f : Z \to \mathbb{R}^\infty$$
is continuous if and only if each of the maps
$$f_i : Z \to \mathbb{R}, \qquad z \mapsto f(z)_i$$
(sending z to the ith component of $f(z)$) is continuous.

2.6 Extra credit (5 points), very hard

Endow $\mathbb{R}^3$ with the standard metric. Let X and Y be subsets of $\mathbb{R}^3$—treat each as a metric space with the inherited metric.

Is it possible that there exists an isometry from X to Y that does *not* extend to an isometry from $\mathbb{R}^3$ to itself?

2.7 Extra credit (5 points, very hard)

Fix $n \geq 1$. Let X be the set of all $n \times n$ matrices with real number entries. If you have taken linear algebra, you know that an element $A \in X$ may be thought of as (uniquely) specifying a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Let us define

$$||A|| = \sup_{v \neq 0 \in \mathbb{R}^n} \frac{|Av|}{|v|}.$$

Here, $|Av|$ refers to the norm—aka length—of the vector Av, and likewise for $|v|$. "Supremum" refers to the supremum over all non-zero vectors $v \in \mathbb{R}^n$; and if you are uncomfortable with the notion of supremum, you may replace "sup" by "max". (In our situation, the supremum is achieved as a maximum.) Let us define

$$d(A, A') = ||A' - A||.$$

Show that this makes X into a metric space.