3 Due Tuesday, September 17

3.1

Which of the following is a true statement? Every correct choice gives +1. If you incorrectly identify a false statement as true, you get -1 point.

1. The sequence $x_j = 1/j$ converges to 0 in $(\mathbb{R}, d_{std})$.

2. The constant sequence $x_j = 0$ converges to 0 in $(\mathbb{R}, d_{std})$.

3. Suppose that two sequences $x_1, x_2, \dots$ and $y_1, y_2, \dots$ both converge to the same point z. Construct a new sequence $z_1, z_2, \dots$ by:

   $$z_i = \begin{cases} x_{(i+1)/2} & i \text{ odd} \\ y_{i/2} & i \text{ even.} \end{cases}$$

   Then the sequence z_i converges to z.

4. Fix sequences $x_1, x_2, \dots$ and $y_1, y_2, \dots$ in $\mathbb{R}$. If both sequences converge, then the sequence

   $$x_1 + y_1, x_2 + y_2, \dots$$

   converges. (Put another way, this is the sequence whose ith term is $x_i + y_i$.)

3.2

Which of the following is a true statement? Every correct choice gives +1. If you incorrectly identify a false statement as true, you get -1 point.

1. For any metric space (X, d), the empty set $\emptyset \subset X$ is always an open set.

2. For any metric space (X, d), the set X is always an open set.

3. For any metric space (X, d), and for any positive real number $r > 0$ and any element $x \in X$, the open ball $\text{Ball}(x, r)$ of radius r centered at x is always an open set.

4. Suppose that x is contained in some open ball $\text{Ball}(x', r')$ centered at some x' and having some radius r'. Then there exists some $r > 0$ so that $\text{Ball}(x, r)$ is contained in $\text{Ball}(x', r')$.

5. If two open balls contains some element $x \in X$, then the two open balls must be equal.

6. Given $x \in X$, there exist only finitely many open balls in X containing x.

7. Every metric space has infinitely many distinct open balls.

3.3

Which of the following is a true statement? Every correct choice gives +1. If you incorrectly identify a false statement as true, you get -1 point.

1. Let $(X, d) = (\mathbb{R}, d_{std})$. The subset $\mathbb{Q} \subset \mathbb{R}$ of all rational numbers is open.

2. For any metric space (X, d), a subset $A \subset X$ either is open, or is the complement of an open set.

3. Let $(X, d) = (\mathbb{R}, d_{std})$, and fix a closed interval $[a, b] \subset X$ with $a < b$. Then the complement of $[a, b]$ is open.

4. Let $(X, d) = (\mathbb{R}^2, d_{std})$, and fix a square $[a, b] \times [a, b] \subset X$ with $a < b$. Then the complement of $[a, b] \times [a, b]$ is open.

3.4 (10 points)

Let (X, d) be a metric space. Recall that a subset $U \subset X$ is called *open* if U can be written as a union of open balls of X.

Prove the following statements:

(a) Let $U_1, \ldots, U_k$ be a finite collection of open sets. Then the intersection

$$U_1 \cap \ldots U_k$$

is an open subset of X.

(b) Consider an arbitrary collection of open sets. We will use the notation

$$\{U_\alpha\}_{\alpha \in \mathcal{A}}$$

to denote this collection. That is, we have some arbitrary set $\mathcal{A}$, and for every $\alpha \in \mathcal{A}$, we can specify some open set U_α. The set above refers to collection of all U_α.

Prove that the union

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$

is open.

3.5 Extra Credit (5 points)

Let X be any finite set. Show that for any metric on X, every subset of X is open.

3.6 Extra Credit (5 points)

Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose X and Y are both finite sets. Show that any bijection from X to Y is continuous.

3.7 Extra Credit (5 points)

Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose X and Y are both finite sets. Show that any bijection from X to Y is continuous.

3.8 Extra Credit (5 points, very hard)

Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \sum_{n=0}^{\infty} (\frac{1}{2})^n \cos((13)^n \pi x).$$

Show that $f(x)$ is continuous. Extra extra credit: Does it have a derivative?