4 Due Tuesday, September 24

4.1 Multiple Choice

A correct response gets you 1 point. If you incorrectly label a false statement as true, you will get -1 points.

Let (X, d_X) be a metric space and let $U \subset X$ be open. Which of the following is true?

(a) For any $x \in U$, there exists some $\delta > 0$ so that $\mathrm{Ball}(x, \delta) \subset U$.

(b) U is a union of some collection of open balls.

(c) U is a union of some finite collection of open balls.

(d) U cannot be written as a union of finitely many open balls.

(e) U is an open ball.

(f) U cannot be the empty set.

(g) U cannot be X itself.

4.2 Multiple Choice

A correct response gets you 1 point. If you incorrectly label a false statement as true, you will get -1 points.

Let (X, d_X) and (Y, d_Y) be metric spaces. Recall from last homework that we can define a metric space structure on $X \times Y$ by declaring

$$d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

Which of the following is true?

(a) If $U \subset X$ and $V \subset Y$ are open subsets, then $U \times V$ is an open subset of $X \times Y$.

(b) Every open subset of $X \times Y$ can be written $U \times V$, where U is open in X and V is open in Y.

(c) If $U \subset X$ is an open subset, then $U \times Y$ is an open subset of $X \times Y$.

4.3 (Mandatory for some) Extra Credit (5 points)

If you got a 6 or below on any of the writing assignments, you must complete this problem. It will be treated as extra credit, so that it cannot bring down your average; but you must submit a solution to this problem.

If you did not get 6 or below, you can still submit this problem; it will be treated as extra credit.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3 & x \leq 0 \\ 10 & x > 0. \end{cases}$$

Consider both the domain and codomain as a metric space using the standard metric on $\mathbb{R}$. Prove that f is not continuous in three ways:

(a) by using the ϵ-δ definition of continuity,

(b) by using the "preimage of an open set is open" criterion of continuity, and

(c) by using the convergent sequence criterion of continuity.

4.4 (10 points)

Let $(X, \mathcal{T}_X)$ be a topological space, and fix a surjection $p : X \to Y$. Let us declare $\mathcal{T}_Y$ to be the collection of those subsets $V \subset Y$ such that $p^{-1}(V)$ is an open subset of X.

We call this the *quotient topology* of Y.

(a) Show that $(Y, \mathcal{T}_Y)$ is a topological space.

(b) Let Z be a topological space. Show that $f : Y \to Z$ is continuous if and only if the composition $f \circ p : X \to Z$ is continuous.

4.5 Extra Credit (5 points)

Let [1] be the set consisting of two elements called 0 and 1. We consider [1] as a topological space by declaring its open sets to be

$$\{0, 1\}, \{1\}, \emptyset.$$

Let X be a topological space, $\mathcal{T}$ its collection of open sets, and $C^0(X, [1])$ the collection of continuous functions from X to $[1]$.

Exhibit a bijection between $\mathcal{T}$ and $C^0(X, [1])$.

4.6 Extra Credit (5 points)

A *norm* on $\mathbb{R}^d$ is a choice of function $N : \mathbb{R}^d \to \mathbb{R}$ such that

1. $N(x) = 0$ if and only if $x = 0$,
2. $N(tx) = |t| \cdot N(x)$ for any $t \in \mathbb{R}$ and any $x \in \mathbb{R}^d$.
3. $N(x + x') \leq N(x) + N(x')$ for any $x, x' \in \mathbb{R}^d$.

Any norm defines a metric by declaring

$$d_N(x, x') = N(x - x').$$

Let N and M be two norms on $\mathbb{R}^d$. Show that the identity function

$$(\mathbb{R}^d, d_N) \to (\mathbb{R}^d, d_M)$$

is continuous.