7 Homework 7: Due Tuesday, October 8

(Don't forget that there are multiple choice problems online.)

7.1 Slow down, this is a neighborhood! (10 points)

Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subset X$ be a subset.

We say that a subset $V \subset X$ is a *neighborhood of* A if and only if there is some open subset U of X such that

$$A \subset U \subset V.$$

(Remember that Hiro uses the convention that $\subset$ need not be a *proper* subset. So for example, $X \subset X$.)

Now let $x \in X$ be an element. We say that a subset $V \subset X$ is a *neighborhood of* x if there exists an open subset $U \subset X$ such that $x \in U$ and $U \subset V$.

Fix a subset $V \subset X$. Prove that the following are equivalent:

(a) V is open.

(b) For every $x \in V$, V is a neighborhood of x.

(c) For every subset $A \subset V$, V is a neighborhood of A.

(d) V is a neighborhood of itself.

7.2 Extra credit (1 point)

Write three examples of topological spaces. Make sure you specify what the topologies are.

7.3 Extra credit (1 point)

Prove your examples are topological spaces.

7.4 Extra credit (1 point)

Write two examples of continuous maps between topological spaces.

7.5 Extra credit (1 point)

Give an example of a metric space and a sequence in that metric space which converges.

7.6 Extra credit (1 point)

Prove your example converges.

7.7 Extra credit (5 points)

Let $\mathbb{R}P^n$ denote the space of lines through the origin in $\mathbb{R}^{n+1}$, topologized by the quotient map $S^n \to \mathbb{R}P^n$. Prove that there exist $n+1$ open subsets $U_1, \ldots, U_{n+1}$ of $\mathbb{R}P^n$, each homeomorphic to $\mathbb{R}^n$, such that

$$U_1 \cup \ldots \cup U_{n+1} = \mathbb{R}P^n.$$

7.8 Extra credit (5 points)

Let $j : S^2 \to S^2$ be a homeomorphism. Is it true that there then exists a homeomorphism $h : \mathbb{R}P^2 \to \mathbb{R}P^2$ for which

$$\begin{tikzcd} S^2 \arrow[r, "j"] \arrow[d, "p"] & S^2 \arrow[d, "p"] \\ \mathbb{R}P^2 \arrow[r, "h"] & \mathbb{R}P^2 \end{tikzcd}$$

commutes? (Here, $p : S^2 \to \mathbb{R}P^2$ is the quotient map.) Put another way, does there always exist an h such that $p \circ j = h \circ p$?

Going the other way: Suppose $h : \mathbb{R}P^2 \to \mathbb{R}P^2$ is a homeomorphism. Does there always exist a homeomorphism $j : S^2 \to S^2$ such that $p \circ j = h \circ p$?