Homework 10: Due Tuesday, November 12

Time guideline, for those who want it: You should be spending at least one hour a day on these homework sets (until you are finished)—writing, multiple choice, and proofs included. So even if you do not finish, you will have spent at least 7 hours on the homework by the time you hand it in.

For this week, I would expect that the proof takes you no more than two hours, and that the multiple choice takes you no longer than two hours. You will get a lot out of the writing assignment if you spend at least three hours on it—this does not mean you write an opus; this means you think a lot and try things out before writing up your thoughts for submission. Even if you finish the proofs and the multiple choice with plenty of time to spare, I strongly encourage you to devote your time to the writing assignment.

Writing

A lot of students have stated something like: "The difference between a closed ball and an open ball is that the closed ball contains its boundary."

For a subset $A \subset \mathbb{R}^2$, how would *you* define the *boundary* of A? Note that I'm not looking for some answer from Wikipedia or the internet—I want you to partake in the intellectual exercise of coming up with a definition that conforms to your intuitions. (Note that nothing is assumed about A except that it is a subset of $\mathbb{R}^2$—it may be open, it may be closed, it may be neither. But if you would like to define the notion of boundary only for certain kinds of subsets A, you may do so; just be explicit about your restrictions.) You may have several definitions you come up with; which ones do you think are best? How do they compare? Are they logically equivalent?

Be precise. If you use vague terms or terms that do not make sense, you will be penalized.

After you make your definition, I'd like to you tell me what your definition yields for the following examples:

1. When $A = \emptyset$.

2. When $A = \mathbb{R}^2$.

3. When A is the closed ball of radius $r > 0$ centered at the origin; that is, when

$$A = \{x \text{ such that } d(x, 0) \leq r\}.$$

4. When A is the open ball of radius $r > 0$ centered at the origin.

5. When A is given by
$$\text{Ball}(0; r) \bigcup \{(r, 0)\}.$$
(That is, when A is the union of an open ball of radius r, together with the point $(r, 0)$ on the x_1-axis.)

Finally, what did you use about $\mathbb{R}^2$? Would your definition make sense for $\mathbb{R}^n$ for other n? Or for an arbitrary metric space? Or an arbitrary topological space?

Multiple Choice

Is online.

10.1 Proof (10 points)

(a) Prove that the quotient of any compact space is compact. That is, if X is a compact topological space, and $\sim$ an equivalence relation on X, then $X/\sim$ is compact (with the quotient topology).

(b) Prove that $\mathbb{R}P^2$ is compact.

(c) Look back on the homework assignment requiring you to prove that $S^1/\sim$ is homeomorphic to $\mathbb{R}^2 \setminus \{(0,0)\}/\sim$. Can some part of that proof be simplified in light of facts you've learned this week and last?

10.2 Extra credit (1 point each)

1. Given an example showing that an arbitrary union of compact sets need not be compact.

2. Give an example of a closed subset of $\mathbb{R}^n$ that is not compact.

3. Give an example of a bounded subset of $\mathbb{R}^n$ that is not compact.

10.3 Extra credit (5 points, very hard)

Is every compact space a quotient of a compact Hausdorff space?