Homework 12: Due Tuesday, November 26

Time recommendation: An hour a day.

Writing

Let $X \subset \mathbb{R}^2$ be the closed ball of radius 1. Introduce an equivalence relation on X as follows:

$$x \sim x' \iff \begin{cases} x, x' \in \partial X & \text{or} \\ x = x'. \end{cases}$$

That is, you leave the points on the interior of X alone, but you collapse all the points on the boundary of X to a single point.

Can you try to draw a picture of $X/\sim$? What does it look like?

More generally, let $U \subset \mathbb{R}^2$ be an open, bounded set. Let $X = \overline{U}$ and define the same equivalence relation as above. Are $X/\sim$ and U^+ (the one-point compactification) related at all? How so? On what properties of U do your assertions make depend?

Just explore. Spend at least half an hour a day making sure you understand what's going on and that you explore.

Multiple Choice

Is online.

Proof (10 points)

Let $n = (0,0,1) \in \mathbb{R}^3$ be the north pole of the sphere. Let

$$p : S^2 \setminus \{n\} \to \mathbb{R}^2, \qquad (x_1, x_2, x_3) \mapsto \frac{1}{1 - x_3}(x_1, x_2)$$

be the stereographic projection. Show that the function

$$S^2 \to (\mathbb{R}^2)^+ = \mathbb{R}^2 \cup \{*\}, \qquad x \mapsto \begin{cases} * & x = n \\ p(x) & x \neq n \end{cases}$$

(the codomain is the one-point compactification of $\mathbb{R}^2$) is a homeomorphism.

Extra Credit (1 point each)

Give examples of the following:

(a) A subset $U \subset \mathbb{R}^2$, with $U \neq \mathbb{R}^2$, that is dense and open.

(b) A subset $U \subset \mathbb{R}^2$ that is dense but not open.

(c) A subset $U \subset \mathbb{R}^2$, with $U \neq \mathbb{R}^2$, that is dense and connected.

(d) A subset $U \subset \mathbb{R}^2$ that is dense and not connected.

Extra Credit (5 points)

For every $n \geq 1$, find a homeomorphism $f_n : S^2 \to S^2$ such that there exists exactly n points $x \in S^2$ satisfying $f_n(x) = x$. (We call such a point a *fixed point* of f).

Extra Credit (10 points)

Is it possible to find a homeomorphism of S^2 with no fixed points?