# 2 Solutions to HW 1

## Multiple choice questions

**Bolded responses are the correct responses.**

### 2.1

Which of the following statements are true?

(a) **$d(x, x') = d(x', x)$. That is, $d$ is *symmetric.* (For this statement to be true, we man that for *any* choice of $x$ and any choice of $x'$, the equality holds.)**

(b) **If $d(x, x') = 0$, then $x = x'$.**

(c) Let $x, x', x''$ be three real numbers. Then $d(x, x') + d(x', x'') \leq d(x, x'')$. (Remember, that this statement is true means that for *any* choice of three real numbers $x$, $x'$, and $x''$, this inequality holds.)

(d) $d(x, x')$ may be negative. (That is, there exists a pair of real numbers $x$ and $x'$ for which $d(x, x')$ is a negative number.)

(e) Let $x, x', x''$ be three real numbers. If $d(x, x') = d(x, x'')$ then $x' = x''$.



### 2.2

Which of the following statements are true?

(a) (Scaling.) For all real numbers $k, x, x'$, we have

$$d(kx, kx') = kd(x, x').$$

(b) **(Translation invariance.) For all real numbers $x, x', x''$, we have**

$$d(x - x'', x' - x'') = d(x, x').$$

(c) For all real numbers $x, x', x''$, we have

$$d(x + x'', x') = d(x, x') + x''.$$

(d) (Skew-symmetry.) $d(x, x') = -d(x', x)$.

## 2.3

Fix an integer $n \geq 1$.[3] We let $X = \mathbb{R}^n$, so that an element $x \in X$ is the data of $n$ real numbers, called the *coordinates* of $x$. We will denote the coordinates of $x$ by

$$(x_1, \ldots, x_n).$$

So for example, $(\pi, e, 13)$ is an element of $\mathbb{R}^3$.

We define

$$d(x, x') = \sqrt{(x_1 - x'_1)^2 + \ldots + (x_n - x'_n)^2}.$$

Which of the following are true?

(a) **For any two $x, x' \in \mathbb{R}^n$, we have $d(x, x') = d(x', x)$. That is, $d$ is *symmetric*.**

(b) **For any two $x, x' \in \mathbb{R}^n$, if $d(x, x') = 0$, then $x = x'$.**

(c) For any three $x, x', x'' \in \mathbb{R}^n$, we have $d(x, x') + d(x', x'') \leq d(x, x'')$.

## 2.4

For $n \geq 0$, we let $0 \in \mathbb{R}^{n+1}$ denote the origin; this is the element whose coordinates are all equal to zero.

We let $S^n$ denote the set of all points $x'$ such that $d(0, x') = 1$.

Which of the following are true?

(a) **($n = 0$.) $S^0$ consists of exactly two points.**

(b) **($n = 1$.) $S^1$ is a circle.**

(c) **($n = 2$.) $S^2$ is a sphere.**

(d) ($n = 3$.) $S^3$ is a cube.



[3] Remember, this means that $n$—in what follows—is any integer greater than or equal to 1.

## Proofs

### 2.5 (10 points)

Fix an infinite sequence of real numbers $x_1, x_2, \ldots,$.

**Definition 2.5.1.** We say that the sequence *converges*, or is *convergent*, if there exists a real number $x$ such that the following holds:

For every $\epsilon > 0$, there exists an integer $N$ such that

$$i > N \implies |x_i - x| < \epsilon.$$

Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and if a sequence $x_1, x_2, \ldots$ converges, then the sequence $f(x_1), f(x_2), \ldots$ also converges.

We will in fact prove more; we will prove that if the sequence $x_1, \ldots$ converges to $x$, then the sequence $f(x_1), \ldots$ converges to $f(x)$.
To do so, given $\epsilon > 0$, we must find a number $N$ such that $i > N$ implies $|f(x_i) - f(x)| < \epsilon$. By continuity, we can find a $\delta$ such that $|x_i - x| < \delta$ implies $|f(x_i) - f(x)| < \epsilon$. By the assumption that $x_1, \ldots$ converges, there is some $N_\delta$ such that $i > N_\delta \implies |x_i - x| < \delta$. We are finished by taking $N = N_\delta$.