4 Solutions to Homework Two

4.1

Which of the following choices of (X, d) is a metric space? Correct responses in blue.

(a) $X = \mathbb{R}$, and $d(x, x') = x$.

(b) $X = \mathbb{R}$, and $d(x, x') = x - x'$.

(c) $X = \mathbb{R}$, and $d(x, x') = |x - x'|$.

(d) $X = \mathbb{R}$, and $d(x, x') = \sqrt{(x' - x)^2}$.

(e) $X = \mathbb{R}^3$ and $d(x, x') = |x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3|$.

(f) $X = \mathbb{R}^3$ and $d(x, x') = |x_1 - x'_1| + |x_2 - x'_2|$.

(g) X is any set, and $d(x, x') = 0$.

(h) X is any set, and

$$d(x, x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x'. \end{cases}$$

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(a) does not define a metric because $d(0, x') = 0$ for any x'. In particular, it does not satisfy property (0) of a metric.
(b) does not satisfy the symmetry property.
(f) also fails property (0). If x and x' are such that $x_1 = x'_1$ and $x_2 = x'_2$ but their third coordinates are unequal, $d(x, x')$ still equals 0.
(g) also fails property (0).
The other choices are metrics we've seen in class.

4.2

Let (X, d_X) and (Y, d_Y) be metric spaces. Which of the following statements is true?

(a) An isometry $f : X \to Y$ is a continuous function.

(b) If $f : X \to Y$ is an isometry, then its inverse function is also an isometry.

(c) If $f : X \to Y$ is continuous, then f is a bijection.

(d) If $f : X \to Y$ is a bijection, then f is an isometry.

(e) Let (Z, d_Z) be another metric space. If $f : X \to Y$ is an isometry and $g : Y \to Z$ is an isometry, then the composition $g \circ f : X \to Z$ is an isometry.

(f) Let (Z, d_Z) be another metric space. If $f : X \to Y$ is continuous and $g : Y \to Z$ is continuous, then the composition $g \circ f : X \to Z$ is continuous.

(a) is something we saw in class.
(b) is true: If f is a bijection, it has a unique inverse we will call g, so $d_Y(y, y') = d_Y(f(g(y)), f(g(y')))$. If f is an isometry, then $d_Y(f(g(y)), f(g(y'))) = d_X(g(y), g(y'))$. Combining these equalities, we see that g is an isometry.
(c) is false because a continuous function need not be a bijection. For example, the constant map $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 3$ is continuous, but clearly neither a surjection nor an injection.
(d) Not every bijection need be an isometry. For example, the identity function from $(\mathbb{R}^n, d_{discrete})$ to $(\mathbb{R}^n, d_{std})$ is a continuous bijection but not an isometry.
(e) This is straightforward: $d_Z(gf(x), gf(x')) = d_Y(f(x), f(x')) = d_X(x, x')$.
(f) This was shown in class, and you can prove it using ϵ-δ definition straightforwardly.

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4.3 (10 points)

Let (X, d_X) and (Y, d_Y) be metric spaces. We endow the product $X \times Y$ with the following metric:

$$d((x, y), (x', y')) = d(x, x') + d(y, y').$$

Prove the following:

(a) The projection map

$$X \times Y \to X, \qquad (x, y) \mapsto x$$

is continuous.

(b) Let Z be a metric space and fix two continuous functions $f : Z \to X$ and $g : Z \to Y$. Then the function

$$f \times g : Z \to X \times Y, \qquad (f \times g)(z) = (f(z), g(z))$$

is continuous.

(c) Let $C^0(Z, X)$ denote the set of continuous functions from Z to X. Prove that the product set

$$C^0(Z, X) \times C^0(Z, Y)$$

is in bijection with the set

$$C^0(Z, X \times Y).$$

(a) This could be proven using ϵ-δ or using the "preimage of an open set is open" criterion. In this case, it turns out that ϵ-δ is easiest. For example, given $\epsilon > 0$, choose $\delta = \epsilon$. (Any number less than or equal to ϵ will do.) For then

$$d((x, y), (x', y')) < \delta \implies d_X(x, x') + d_Y(y, y') < \delta$$

but $d_Y(y, y') \geq 0$, so this means $d_X(x, x') < \delta = \epsilon$.

(b) Given ϵ, choose positive numbers ϵ_1, ϵ_2 so that $\epsilon_1 + \epsilon_2 \leq \epsilon$. Then there exist δ_1, δ_2 so that

$$d_Z(z, z') < \delta_1 \implies d_X(f(z), f(z')) < \epsilon_1$$

and

$$d_Z(z, z') < \delta_2 \implies d_Y(g(z), g(z')) < \epsilon_2$$

by the continuity of f and of g. In particular, choose any δ that is less than both δ_1 and δ_2. Then

$$d_Z(z, z') < \delta \implies d_X(f(z), f(z')) + d_Y(g(z), g(z')) < \epsilon_1 + \epsilon_2 \leq \epsilon$$

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(c) Let us give names to the projections. Let $p_X : X \times Y \to X$ be the map sending $(x, y) \mapsto x$. Likewise for p_Y. Note these are both continuous by part (a).

Given an element (f, g) in the first set, define $\alpha(f, g)$ to be the function sending $z \in Z$ to $(f(z), g(z))$. Likewise, given an element h in the latter set, define $\beta(h)$ to be the pair $(p_X \circ h, p_Y \circ h)$.

First we claim that α and β are mutually inverse functions. To see this, we claim that the first coordinate of $\beta \circ \alpha(f, g)$—that is, $p_X \circ \alpha(f, g)$—is equal to f. This is because for any $z \in Z$, we have

$$p_X \circ \alpha(f, g)(z) = p_X(f(z), g(z)) = f(z).$$

Likewise, the second coordinate of $\beta \circ \alpha(f, g)$ is g. This shows that

$$\beta \circ \alpha(f, g) = (f, g).$$

We now show that $\alpha \circ \beta(h) = h$. This is because

$$(\alpha \circ \beta(h))(z) = (\alpha(p_X \circ h, p_Y \circ h))(z) = (p_X \circ h(z), p_Y \circ h(z)) = h(z).$$

Now all that remains to check is that (i) $\alpha(h)$ is a continuous function, and (ii) both components of $\beta(f, g)$ are continuous. (This is to make sure that α indeed has image in $C^0(Z, X \times Y)$ and that β indeed has image in $C^0(Z, X) \times C^0(Z, Y)$.) (i) follows from part (b), while (ii) follows from part (a) and the in-class result that a composition of continuous functions is continuous.

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4.4 Extra credit (10 points)

Write out a complete explanation for why each of the multiple choice options from Homework One are either true, or false.

These explanations must be succinct and correct for full credit.

4.5 Extra credit (5 points)

Let $\mathbb{R}^\infty$ denote the set of all infinite sequences

$$x = (x_1, x_2, \dots)$$

for which all but finitely many x_i equal 0. You do not need to prove this, but the function

$$d(x, x') = \sum_{i=1}^{\infty} |x'_i - x_i|$$

renders $\mathbb{R}^\infty$ a metric space.

Show that for any metric space Z, a function

$$f : Z \to \mathbb{R}^\infty$$

is continuous if and only if each of the maps

$$f_i : Z \to \mathbb{R}, \qquad z \mapsto f(z)_i$$

(sending z to the ith component of $f(z)$) is continuous.

This extra credit problem is incorrect—indeed, a function f is continuous if each f_i is continuous. But the converse is not true.
For example, define f to be a function from $\mathbb{R}$ to $\mathbb{R}^\infty$ such that

$$f_i(t) = i \cdot t.$$

For any $\epsilon > 0$, and for any $\delta > 0$, there is some integer $i > 0$ so that $i\delta/2 > \epsilon$. Thus even if x and $x' \in \mathbb{R}$ are within distance δ of each other, if their ith coordinate is larger than $\delta/2$ apart, the image will be distance $> \epsilon$ apart.

4.6 Extra credit (5 points), very hard

Endow $\mathbb{R}^3$ with the standard metric. Let X and Y be subsets of $\mathbb{R}^3$—treat each as a metric space with the inherited metric.

Is it possible that there exists an isometry from X to Y that does *not* extend to an isometry from $\mathbb{R}^3$ to itself?

4.7 Extra credit (5 points, very hard)

Fix $n \geq 1$. Let X be the set of all $n \times n$ matrices with real number entries. If you have taken linear algebra, you know that an element $A \in X$ may be thought of as (uniquely) specifying a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

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Let us define

$$||A|| = \sup_{v \neq 0 \in \mathbb{R}^n} \frac{|Av|}{|v|}.$$

Here, $|Av|$ refers to the norm—aka length—of the vector Av, and likewise for $|v|$. "Supremum" refers to the supremum over all non-zero vectors $v \in \mathbb{R}^n$; and if you are uncomfortable with the notion of supremum, you may replace "sup" by "max". (In our situation, the supremum is achieved as a maximum.) Let us define

$$d(A, A') = ||A' - A||.$$

Show that this makes X into a metric space.

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