3 Solutions to Homework 3

3.1

Which of the following is a true statement? Every correct choice gives +1. If you incorrectly identify a false statement as true, you get -1 point. correct responses in blue

1. The sequence $x_j = 1/j$ converges to 0 in $(\mathbb{R}, d_{std})$.

2. The constant sequence $x_j = 0$ converges to 0 in $(\mathbb{R}, d_{std})$.

3. Suppose that two sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ both converge to the same point z. Construct a new sequence $z_1, z_2, \ldots$ by:

$$z_i = \begin{cases} x_{(i+1)/2} & i \text{ odd} \\ y_{i/2} & i \text{ even.} \end{cases}$$

   Then the sequence z_i converges to z.

4. Fix sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ in $\mathbb{R}$. If both sequences converge, then the sequence

$$x_1 + y_1, x_2 + y_2, \ldots$$

   converges. (Put another way, this is the sequence whose ith term is $x_i + y_i$.)

(1) is true. Given any ϵ, there exists an integer N so that $1/N < \epsilon$. Moreover, if $i > i'$, we have that $1/i' < 1/i$.

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(2) is true. For any i, $|x_i - 0| = 0$. In particular, for any $\epsilon > 0$ and for any i (regardless of choice of N), we have that $|x_i - 0| < \epsilon$.

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(3) is true. For example, given ϵ, we have N_x and N_y such that $i > N_x \implies |x_i - z| < \epsilon$ and likewise for $j > N_y \implies |y_j - z| < \epsilon$. Taking $N > 2N_x$ and $N > 2N_y$, we see that z_i converges to z. If the two sequences did not converge to the same point, it would be not true. As an example, you can take $x_1, \ldots$ to be the constant sequence with value 0, and you can take $y_1, \ldots$ to be the constant sequence with value 1. This does not converge to any value, as given any number A and any $\epsilon < 1/2$, either i odd or i even will guarantee that $|z_i - A|$ is larger than ϵ.

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(4) This is true. In fact, let x and y be the values to which the sequences $x_1, \ldots$ and $y_1, \ldots$ converge, respectively. I claim that the sequence $x_1 + y_1, \ldots$ converges to $x + y$.

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To see this, given $\epsilon > 0$, choose N_x and N_y so that

$$i > N_X \implies |x_i - x| < \epsilon/2, \qquad j > N_Y \implies |y_j - y| < \epsilon/2.$$

Then choose N to be any number larger than N_X and N_Y. We see that

$$i > N \implies |x_i + y_i - (x + y)| \leq |x_i - x| + |y_i - y| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

3.2

Which of the following is a true statement? Every correct choice gives +1. If you incorrectly identify a false statement as true, you get -1 point.

1. For any metric space (X, d), the empty set $\emptyset \subset X$ is always an open set.
2. For any metric space (X, d), the set X is always an open set.
3. For any metric space (X, d), and for any positive real number $r > 0$

and any element $x \in X$, the open ball $\text{Ball}(x, r)$ of radius r centered at x is always an open set.

4. Suppose that x is contained in some open ball $\text{Ball}(x', r')$ centered at some x' and having some radius r'. Then there exists some $r > 0$ so that $\text{Ball}(x, r)$ is contained in $\text{Ball}(x', r')$.

5. If two open balls contains some element $x \in X$, then the two open balls must be equal.

6. Given $x \in X$, there exist only finitely many open balls in X containing x.

7. Every metric space has infinitely many distinct open balls.

1. is true. We saw this in class.

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2. is true. We also saw this in class.

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3. is true by definition of open set. We saw this in class.

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4. This is the centering lemma. It is true, we saw this in class.

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5. This is false. For example, both $\text{Ball}(0; 1)$ and $\text{Ball}(0; 2)$ in $(\mathbb{R}, d_{std})$ contain the origin but they are not equal balls.

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6. This is false in general. For example, if $x = 0$ is the origin of $\mathbb{R}$, one can choose infinitely many radii r so that $\text{Ball}(x; r)$ contains x.

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7. This is false. For example, one could take a metric space X to consist of a finite set. There are only finitely many subsets of X, and in particular, only finitely many open balls.

3.3

Which of the following is a true statement? Every correct choice gives +1. If you incorrectly identify a false statement as true, you get -1 point.

1. Let $(X, d) = (\mathbb{R}, d_{std})$. The subset $\mathbb{Q} \subset \mathbb{R}$ of all rational numbers is open.

2. For any metric space (X, d), a subset $A \subset X$ either is open, or is the complement of an open set.

3. Let $(X, d) = (\mathbb{R}, d_{std})$, and fix a closed interval $[a, b] \subset X$ with $a < b$. Then the complement of $[a, b]$ is open.

4. Let $(X, d) = (\mathbb{R}^2, d_{std})$, and fix a square $[a, b] \times [a, b] \subset X$ with $a < b$. Then the complement of $[a, b] \times [a, b]$ is open.

1. is false. Fix a rational number—for example, 0. Then an open ball of radius r centered at 0 is an open interval of the form $(-r, r)$. Such an open interval contains a non-rational number, and in particular, $(-r, r)$ is not contained in $\mathbb{Q}$.

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2. is false. An example is $A = \mathbb{Q}$. We saw above that $\mathbb{Q}$ is not open. The complement of $\mathbb{Q}$ is the set of irrational numbers; this is also not open. This is because any open interval contains (infinitely many!) rational numbers, so no open interval is contained in the set of irrational numbres.

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3. is true. For any $x \notin [a, b]$, there is some minimal distance from x to the interval $[a, b]$. Letting r be this distance—explicitly, $r = \min\{|x-a|, |x-b|\}$—we have that the open ball of radius r centered at x is contained in $\mathbb{R} \setminus [a, b]$.

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4. is true. Given a point $x = (x_1, x_2)$ in the complement of the square, let $r = \min\{|x_1 - a|, |x_2 - a|, |x_1 - b|, |x_2 - b|\}$. Then the open ball of radius r about x is contained in the complement.

3.4 (10 points)

Let (X, d) be a metric space. Recall that a subset $U \subset X$ is called *open* if U can be written as a union of open balls of X.

Prove the following statements:

(a) Let $U_1, \ldots, U_k$ be a finite collection of open sets. Then the intersection

$$U_1 \cap \ldots U_k$$

is an open subset of X.

(b) Consider an arbitrary collection of open sets. We will use the notation

$$\{U_\alpha\}_{\alpha \in \mathcal{A}}$$

to denote this collection. That is, we have some arbitrary set $\mathcal{A}$, and for every $\alpha \in \mathcal{A}$, we can specify some open set U_α. The set above refers to collection of all U_α.

Prove that the union

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$

is open.

(a) Let x be in the intersection, so that x is an element of each $U_1, \ldots, U_k$. For every i between 1 and k, we know that there is some radius r_i so that

$$\mathrm{Ball}(x; r_i) \subset U_i.$$

(This follows from the definition of open set and the centering lemma.) Note that if $r_i > r_j$, we have

$$\mathrm{Ball}(x; r_j) \subset \mathrm{Ball}(x; r_i).$$

In particular, choosing r to be (the minimum, or) less than any of the $r_1, \ldots, r_k$, we see that $\mathrm{Ball}(x; r)$ is contained in each of the $U_1, \ldots, U_k$. In particular, the ball is on the intersection.

Thus we've shown that for any $x \in U_1 \cap \ldots \cap U_k$, there is some small $r > 0$ for which $\mathrm{Ball}(x; r)$ is contained in $U_1 \cap \ldots \cap U_k$. This shows that the intersection $U_1 \cap \ldots \cap U_k$ is open.

(b) Write each U_α as a union of open sets; that is, for each α, let $\mathcal{B}_\alpha$ be some set, and let

$$U_\alpha = \bigcup_{b \in \mathcal{B}_\alpha} \mathrm{Ball}(x_b; r_b).$$

Then

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = \bigcup_{\alpha \in \mathcal{A}} \bigcup_{b \in \mathcal{B}_\alpha} \mathrm{Ball}(x_b; r_b).$$

Thus $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is a union of open balls, hence it is open.

3.5 Extra Credit (5 points)

Let X be any finite set. Show that for any metric on X, every subset of X is open.

Consider the set of all pairs $x, x' \in X$ for which $x \neq x'$. Then

$$\{d(x, x')\}_{x \neq x'}$$

is a finite collection of positive real numbers. Choose some $r > 0$ less than any number in this collection. Note that $\text{Ball}(x; r)$ consists of only x, but is open (because it is an open ball). Since any subset of X is a union of the singleton sets $\{x\}$, any subset of X is a union of open balls.

3.6 Extra Credit (5 points)

Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose X and Y are both finite sets. Show that any bijection from X to Y is continuous.

This is easy given the previous problem. In fact, any function $f : X \to Y$ is continuous. This is because given any $V \subset Y$, the preimage $f^{-1}(V)$ is open in X by the previous problem.

3.7 Extra Credit (5 points)

Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose X and Y are both finite sets. Show that any bijection from X to Y is continuous.

This is a duplicate problem.

3.8 Extra Credit (5 points, very hard)

Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \sum_{n=0}^{\infty} (\frac{1}{2})^n \cos((13)^n \pi x).$$

Show that $f(x)$ is continuous. Extra extra credit: Does it have a derivative?