4 Solutions to Homework 4

4.1 Multiple Choice

A correct response gets you 1 point. If you incorrectly label a false statement as true, you will get -1 points.

Let (X, d_X) be a metric space and let $U \subset X$ be open. Which of the following is true?

(a) For any $x \in U$, there exists some $\delta > 0$ so that $\text{Ball}(x, \delta) \subset U$.

(b) U is a union of some collection of open balls.

(c) U is a union of some finite collection of open balls.

(d) U cannot be written as a union of finitely many open balls.

(e) U is an open ball.

(f) U cannot be the empty set.

(g) U cannot be X itself.

(a) is true. We saw this in class.

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(b) is true. This is the definition of open set in a metric space.

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(c) is false. For example, let $(X, d_X) = (\mathbb{R}, d_{std})$ and let $U = X \setminus \mathbb{Z}$—that is, all real numbers that are not integers. This is open—as it is a union of open intervals—but it cannot be written as a union of finitely many open intervals.

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(d) is false. For example, U itself could be an open ball—in which case it is the union of a single open ball.

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(e) is false. For example, U can be the example above of $\mathbb{R} \setminus \mathbb{Z}$.

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(f) is false. We saw in class that the empty set is an example of an open set.

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(g) is false. We saw in class that X is an example of an open set.

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4.2 Multiple Choice

A correct response gets you 1 point. If you incorrectly label a false statement as true, you will get -1 points.

Let (X, d_X) and (Y, d_Y) be metric spaces. Recall from last homework that we can define a metric space structure on $X \times Y$ by declaring

$$d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

Which of the following is true?

(a) If $U \subset X$ and $V \subset Y$ are open subsets, then $U \times V$ is an open subset of $X \times Y$.

(b) Every open subset of $X \times Y$ can be written $U \times V$, where U is open in X and V is open in Y.

(c) If $U \subset X$ is an open subset, then $U \times Y$ is an open subset of $X \times Y$.

(a) is true. To see this, choose $(x, y) \in U \times V$. We know there are positive numbers δ_x, δ_y such that $\text{Ball}(x; \delta_x) \subset U$ and $\text{Ball}(y; \delta_y) \subset V$ (because U and V are open). Let δ be less than both δ_x and δ_y; for example, set $\delta = \min\{\delta_x, \delta_y\}$. Then I claim

$$\text{Ball}((x, y); \delta) \subset U \times V.$$

This is because if (x', y') is distance δ away from (x, y), this means that

$$d_X(x, x') < d_X(x, x') + d_Y(y, y') = d((x, y), (x', y')) < \delta \leq \delta_x.$$

and likewise, $d_Y(y, y') < \delta_y$. Thus $x' \in U$ and $y' \in V$, meaning $(x', y') \in U \times V$.
(b) is false. For example, let $X = Y = \mathbb{R}$, in which case the metric on $X \times Y$ is the taxicab metric. Then the complement of a square $[a, b] \times [a, b]$ is open in $X \times Y$, but this complement cannot be written as a product of two open sets.
(c) is true by part (a), by setting $V = Y$.

4.3 (Mandatory for some) Extra Credit (5 points)

If you got a 6 or below on any of the writing assignments, you must complete this problem. It will be treated as extra credit, so that it cannot bring down your average; but you must submit a solution to this problem.

If you did not get 6 or below, you can still submit this problem; it will be treated as extra credit.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3 & x \leq 0 \\ 10 & x > 0. \end{cases}$$

Consider both the domain and codomain as a metric space using the standard metric on $\mathbb{R}$. Prove that f is not continuous in three ways:

(a) by using the ϵ-δ definition of continuity,

(b) by using the "preimage of an open set is open" criterion of continuity, and

(c) by using the convergent sequence criterion of continuity.

In what follows, we will use the notation $(X, d_X) = (Y, d_Y) = (\mathbb{R}, d_{std})$ to distinguish the domain from the codomain.

(a) Let $x = 0$ and choose ϵ to be any positive real number less than $10 - 3 = 7$. Then for any $\delta > 0$, note that there always exist an $x' > x$ for which $d(x, x') < \delta$. Moreover, for such x', we have

$$d_Y(f(x), f(x') = |10 - 3| = 7.$$

In particular, regardless of $\delta > 0$, we have that $d_Y(f(x), f(x')) > \epsilon$ even if $d_X(x, x') < \delta$. This shows f is not continuous.

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(b) Let $V = (3 - r, 3 + r)$ be an open interval of width $2r$ centered at 3. This is in particular an open ball of radius r in Y, so is an open subset of Y. We choose $r < 7$. Then the preimage $f^{-1}(V)$ is the set of all x such that $x \leq 0$. $f^{-1}(V)$ is not an open subset of $X = \mathbb{R}$ because no open ball of positive radius centered at $x = 0$ is contained in $f^{-1}(V)$. (Any open ball would contain some element larger than x, but no such element is in $f^{-1}(V)$.) Thus there is some open set in Y whose preimage under f is not open; this shows that f is not continuous.

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(c) Let $x_1, x_2, \ldots$ be a sequence converging to $x = 0$ and "sandwiching" x from both sides. For concreteness, let us take

$$x_k = (-1)^k \frac{1}{k} = \begin{cases} -1/k & k \text{ odd} \\ 1/k & k \text{ even} \end{cases}.$$

Then $f(x_1), f(x_2), \ldots$ is the sequence given by

$$f(x_k) = \begin{cases} 3 & k \text{ odd} \\ 10 & k \text{ even.} \end{cases}$$

This sequence does not converge. We have exhibited an example of sequence converging in X such that f of that sequence does not converge in Y; hence f is not continuous.

Another proof: Recall that for f to be a continuous function between metric spaces, it must not only preserve convergent sequences—we have that if $x_1, x_2, \ldots$ converges to x, then $f(x_1), f(x_2), \ldots$ must converge to $f(x)$. One can take $x_1, x_2, \ldots$ to be the sequence $x_k = 1/k$, which converges to $x = 0$. But then $f(x_1), f(x_2)$ is the constant sequence $f(x_k) = 10$, but this does not converge to $f(x) = 3$.

4.4 (10 points)

Let $(X, \mathcal{T}_X)$ be a topological space, and fix a surjection $p : X \to Y$. Let us declare $\mathcal{T}_Y$ to be the collection of those subsets $V \subset Y$ such that $p^{-1}(V)$ is an open subset of X.

We call this the *quotient topology* of Y.

(a) Show that $(Y, \mathcal{T}_Y)$ is a topological space.

(b) Let Z be a topological space. Show that $f : Y \to Z$ is continuous if and only if the composition $f \circ p : X \to Z$ is continuous.

(a) First, note that both $\emptyset$ and Y are in $\mathcal{T}_Y$—this is because $p^{-1}(\emptyset) = \emptyset$ is open in X and $p^{-1}(Y) = X$ is open in X (by definition of topological space).[4]
Next, if $V_1, \ldots, V_k$ is a finite collection of subsets of Y such that $f^{-1}(V_i)$ is open in X for all $i = 1, \ldots, k$, we note that

$$f^{-1}(V_1 \cap \ldots \cap V_k) = f^{-1}(V_1) \cap \ldots f^{-1}(V_k).$$

The righthand side is a finite intersection of open subsets of X, hence is an open subset of X. This shows that $V_1 \cap \ldots \cap V_k$ is in $\mathcal{T}_Y$.
Likewise, if $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ is an arbitrary collection of elements in $\mathcal{T}_Y$, we have that

$$f^{-1}(\bigcup_{\alpha \in \mathcal{A}} V_\alpha) = \bigcup_{\alpha \in \mathcal{A}} f^{-1}(V_\alpha)$$

(the preimage of a union is the union of the preimages) and the righthand side is open in X because it is a union of open sets of X. This shows that $\bigcup_{\alpha \in \mathcal{A}} V_\alpha$ is in $\mathcal{T}_Y$. We are finished.
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(b) $f \circ p$ continuous $\implies$ f continuous: Suppose $f \circ p : X \to Z$ is continuous. Let $W \subset Z$ be open. Then $V = f^{-1}(W) \subset Y$ is a subset such that

$$p^{-1}(V) = p^{-1}(f^{-1}(W)) = (f \circ p)^{-1}(W),$$

and the rightmost set of this equality is open because $f \circ p$ is continuous. Because $p^{-1}(V)$ is open, we conclude that $V = f^{-1}(W)$ is open. This shows f is continuous.
f continuous $\implies$ $f \circ p$ continuous: Now suppose f is continuous. We first claim that p is continuous—this is because $V \subset Y$ is open if and only if $p^{-1}(V)$ is open. By a result from class, a composition of continuous maps is continuous, so $f \circ p$ is continuous. We are finished.

4.5 Extra Credit (5 points)

Let [1] be the set consisting of two elements called 0 and 1. We consider [1] as a topological space by declaring its open sets to be

$$\{0, 1\}, \{1\}, \emptyset.$$

Let X be a topological space, $\mathcal{T}$ its collection of open sets, and $C^0(X, [1])$ the collection of continuous functions from X to [1].

Exhibit a bijection between $\mathcal{T}$ and $C^0(X, [1])$.

Solution omitted.

4.6 Extra Credit (5 points)

A *norm* on $\mathbb{R}^d$ is a choice of function $N : \mathbb{R}^d \to \mathbb{R}$ such that

1. $N(x) = 0$ if and only if $x = 0$,
2. $N(tx) = |t| \cdot N(x)$ for any $t \in \mathbb{R}$ and any $x \in \mathbb{R}^d$.
3. $N(x + x') \leq N(x) + N(x')$ for any $x, x' \in \mathbb{R}^d$.

Any norm defines a metric by declaring

$$d_N(x, x') = N(x - x').$$

Let N and M be two norms on $\mathbb{R}^d$. Show that the identity function

$$(\mathbb{R}^d, d_N) \to (\mathbb{R}^d, d_M)$$

is continuous.

Solution omitted.