5 HW 5 Solutions

5.1 Multiple choice.

You get $+1$ for every response correctly chosen, and -1 for every response you incorrectly choose.

Based on what we have defined in this class, which of the following notions make sense?

(a) For a function between two metric spaces to be continuous.

(b) For a function between two topological spaces to be continuous.

(c) For a function between two metric spaces to be continuous.

(d) For a function between two topological spaces to be convergent.

(e) For a function between two topological spaces to be open.

(f) For an open subset of a topological space to be convergent.

(g) For a subset of a metric space to be open.

(h) For a subset of a topological space to be open.

(i) For a sequence in a metric space to be convergent.

(j) For a sequence between two metric spaces to be convergent.

(k) For a sequence between two metric spaces to be continuous.

(l) For a topological space to be convergent.

(m) For an element of a metric space to be convergent.

(n) For an element of a topological space to be continuous.

Answers: (a), (b), (c), *(e) has not yet been defined, but it does turn out to make sense in the literature*, (g), (h), (i).

5.2 Multiple choice.

You get +1 for every response correctly chosen, and −1 for every response you incorrectly choose.

Let X be a set and $E \subset X \times X$ an equivalence relation. Which of the following is true?

(a) If $A, A' \subset X$ are two equivalence classes, then either $A \cap A' = \emptyset$ or $A = A'$.

(b) x and x' are in the same equivalence class if and only if $x \sim x'$.

(c) $X/\sim$ is empty if and only if X is empty.

(d) The function $X \to X/\sim$ is an injection.

(e) The function $X \to X/\sim$ is a surjection.

(a) is true. Suppose $A \cap A' = \emptyset$. Since A and A' are non-empty by definition of equivalence class, we have that A does not equal A'. Now suppose that $A \cap A' \neq \emptyset$, and let $x \in A \cap A'$. Then $A = [x]$ and $A' = [x]$, so $A = A'$.
(b) is true. Suppose x, x' are both in an equivalence class A. One of the properties of an equivalence class is that then $x \sim x'$. On the other hand, another property of equivalence class is that if $x \in A$ and $x' \sim x$, then $x' \in A$. This proves the converse.
(c) This is true. $X/\sim$ is the set of equivalence classes; so if $X/\sim$ is empty, X has no equivalence classes; but by reflexivity, every element of X defines some equivalence class, so if X has no equivalence classes, X must be empty. On the other hand, if X is empty, $X/\sim$ is empty because X has no equivalence classes. (By definition, an equivalence class must be non-empty.)
(d) This is false in general. For example, let $E = X \times X$.
(e) This is true. Every equivalence class $[x]$ is the image of an element $x \in X$.

5.3 Proof (10 points)

Let $(X, \mathcal{T}_X)$ be a topological space. We say that $(X, \mathcal{T}_X)$ is *Hausdorff* if and only if the following holds:

For any $x, x' \in X$ with $x \neq x'$, there exist open sets $U, U' \subset X$ such that

1. $x \in U$ and $x' \in U'$, and
2. $U \cap U' = \emptyset$.

(a) Suppose that d_X is a metric on X, and $\mathcal{T}_X$ is the topology induced by d_X. (So $U \in \mathcal{T}_X$ if and only if U is a union of open balls.) Prove that $(X, \mathcal{T}_X)$ is Hausdorff. (That is, any metric space is Hausdorff.)

(b) Let $(X, d_X) = (\mathbb{R}, d_{std})$ and consider the following equivalence relation on $\mathbb{R}$: We say that $x \sim x'$ if and only if there is a *non-zero* real number t such that $tx = x'$. Show that $X/\sim$ is not Hausdorff.

(a) Since $x \neq x'$, we know $d(x, x') \neq 0$. We showed in class that $d(x, x') \geq 0$ in general, so $d(x, x') > 0$. Let $r = d(x, x')/2$. Then the open balls $U = \text{Ball}(x, r)$ and $U' = \text{Ball}(x', r)$ are both open subsets of X. On the other hand, they have no intersection. Proof by contradiction: If $z \in U \cap U'$, then

$$d(x, z) + d(z, x') \geq d(x, x') = 2r$$

but we know that $d(x, z) + d(z, x') < 2r$.
(b) If x, x' are non-zero, then $t = x'/x$ is a non-zero real number. So $\sim$ has two equivalence classes: $[0]$ and $[1]$.
Thus $X/\sim$ has exactly two elements, $[0]$ and $[1]$. The only subsets of $X/\sim$ that contain $[0]$ are given by

$$\{[0]\} \text{ and } \{[0], [1]\} = X.$$

But $\{[0]\}$ is not open, because its preimage is in $\mathbb{R}$ i not open. (The preimage is the set $\{0\} \subset \mathbb{R}$, and this is not open because it contains no open ball of positive radius, for example). Thus the only open set of X that contains $[0]$ is X itself.
In particular, if any open set $U' \subset X; \sim$ contains $[1]$, its intersection with X is non-empty.
This shows that $X/\sim$ is not Hausdorff.

5.4 Extra Credit (5 points)

Give an example of two topological spaces X and Y, together with a continuous surjection $f : X \to Y$, where Y does *not* have a quotient topology induced by any equivalence relation on X.

5.5 Extra Credit (5 points)

True or false: If X and Y are Hausdorff topological spaces, then their product is Hausdorff. You get no credit unless you justify your answer.

5.6 Extra Credit (5 points)

True or false: Every non-Hausdorff topological space is the quotient of a Hausdorff topological space.