6 HW 6 Solution

Set-up: Let $\mathbb{R}^2\setminus\{(0,0\}$ denote the set of points in $\mathbb{R}^2$ that are not the origin. Define an equivalence relation by declaring that

$$(x_1,x_2)\sim(x_1',x_2')$$

if and only if there exists a non-zero real number t such that

$$tx_1 = x_1' \qquad \text{and} \qquad tx_2 = x_2'.$$

We let

$$X := (\mathbb{R}^2\setminus\{(0,0\})/\sim$$

denote the quotient space. (So X is given the quotient topology.)

On the other hand, define an equivalence relation $\sim_{S^1}$ on S^1 by declaring that

$$(x_1,x_2)\sim_{S^1}(x_1',x_2')$$

if and only if one of the following holds:

$$(x_1,x_2)=(x_1',x_2') \qquad \text{or} \qquad (x_1,x_2)=(-x_1',-x_2').$$

We let

$$Y := S^1/\sim_{S^1}.$$

Problem: Exhibit a homeomorphism from X to Y.
(This verifies that the two topologies we have put on $\mathbb{R}P^1$ are equivalent.)

There are many possible solutions. Here is one:
Define $g : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ to be the obvious inclusion. (So for example, $g(x) = x$.)

Proposition. g is continuous. Proof. Any open subset U of $\mathbb{R}^2 \setminus \{(0,0)\}$ is given as $W \cap \mathbb{R}^2 \setminus \{(0,0)\}$ for some open subset $W \subset \mathbb{R}^2$ (by definition of subspace topology). On the other hand, and $g^{-1}(U) = W \cap S^1$, which is open by the definition of subspace topology (applied to S^1). (End of proof of proposition.)

.

We know from class that the quotient map $q : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}/\sim = X$ is continuous. Thus the composition

$$q \circ g : S^1 \to X$$

is continuous (because the composition of continuous maps is continuous).

.

In what follows, let us write $[x]_{S^1}$ to denote an element of $Y = S^1/\sim_{S^1}$, while we will write $[x]$ to denote an element of X. Consider the function $g' : Y \to X$ given by $[x]_{S^1} \mapsto [x]$. This is well-defined because if $x \sim_{S^1} x'$, then $x = \pm x'$, hence $x \sim x'$.

.

Lemma. g' is continuous. Proof: Setting $q_{S^1} : S^1 \to S^1/\sim_{S^1}$ to be the quotient map, we have

$$q \circ g = g' \circ q_{S^1}.$$

In homework, we saw that $g' \circ q_{S^1}$ is continuous if and only if g' is. But we saw above that $q \circ g$ is continuous; thus g' is continuous. (End of proof.)

.

Lemma. g' is a bijection. Proof: If $g'([x]_{S^1}) = g'([x']_{S^1})$, then $[x] = [x']$, so $x' = tx$ for some $t \neq 0$. But x and x' are points on S^1, so x' and x are vectors of the same norm; this means t must equal 1 or -1. Thus $x \sim_{S^1} x'$, and $[x]_{S^1} = [x']_{S^1}$. Thus g' is an injection. On the other hand, given any element $[x]$, we know let $t = d(x,0)$ be the norm of x. Then $x/t \in S^1$, so $g([x/t]_{S^1}) = [x]$. this shows g' is a surjection. (End of proof.)

.

Continued on next page.

Now define $f : \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$ as follows:

$$f(x) = x/d(0,x).$$

That is, f sends x to the element of $\mathbb{R}^2$ obtained by scaling x by the inverse of the length of x. (Note that the length of x is not zero by assumption on the domain.)

(It is straightforward to verify that $f(x)$ is indeed on the circle.)

Lemma. f is continuous. Proof: We will verify the ϵ-δ criterion. Fix x in the domain and fix $\epsilon > 0$. Without loss of generality, we will assume that ϵ is very small. Then $\text{Ball}(f(x), \epsilon)$ is an open arc on S^1—i.e., it consists of points of the form $(\cos\theta, \sin\theta)$ where θ is contained in some open interval of $\mathbb{R}$ (of length less than, say, π). The preimage of this open arc under f is the subset U of all elements in $\mathbb{R}^2 \setminus \{(0,0)\}$ that can be written as $(r\cos\theta, r\sin\theta)$ for $r > 0$ and for θ in the just-mentioned interval. This U contains an open ball of radius $\delta = \epsilon \cdot |x|$, centered at x. (This is the definition of ϵ if $|x| = 1$; otherwise, scale.) End of proof.

Now define $f' : X \to Y$ just as we defined g'; it is straightforward to verify that f' is continuous (using that f is continuous; just as we showed g' is continuous using g) and that f' is the inverse to g'.