7 HW 7 Solutions

(Don't forget that there are multiple choice problems online.)

7.1 Slow down, this is a neighborhood! (10 points)

Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subset X$ be a subset.

We say that a subset $V \subset X$ is a *neighborhood of A* if and only if there is some open subset U of X such that

$$A \subset U \subset V.$$

(Remember that Hiro uses the convention that $\subset$ need not be a *proper* subset. So for example, $X \subset X$.)

Now let $x \in X$ be an element. We say that a subset $V \subset X$ is a *neighborhood of x* if there exists an open subset $U \subset X$ such that $x \in U$ and $U \subset V$.

Fix a subset $V \subset X$. Prove that the following are equivalent:

(a) V is open.

(b) For every $x \in V$, V is a neighborhood of x.

(c) For every subset $A \subset V$, V is a neighborhood of A.

(d) V is a neighborhood of itself.

$(a) \implies (b)$. Because V is open, we can set $U = V$.
$(b) \implies (c)$. Given $A \subset V$ and for every $x \in A$, let $U_x \subset V$ be the open subset showing that V is a neighborhood of x. Then set $U = \bigcup_{x \in A} U_x$. This U is open because it is a union of open subsets U_x, and we clearly have $A \subset U$ and $U \subset V$.
$(c) \implies (d)$. Take $A = V$.
$(d) \implies (a)$. We know where exists an open subset U satisfying $V \subset U \subset V$. But $V \subset U$ and $U \subset V$ means $U = V$.