Homework 8 Solutions

8.1 Proof (10 points)

tLet $f : X \to Y$ be continuous. Show that any compact subspace of X is sent to a compact subspace of Y.

We must show that if $B \subset X$ is compact, then $f(B)$ is compact.
(Like many problems, we will not need to use the hypothesis (that A is compact) until a few lines in.)
Fix an open cover $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ of $f(B)$. Our goal is to exhibit a finite subcover. By definition of subset topology, for every $\alpha \in \mathcal{A}$, there exists an open set $W_\alpha \subset Y$ such that $W_\alpha \cap f(A) = V_\alpha$. Because $f : X \to Y$ is continuous, we note that $f^{-1}(W_\alpha) \subset X$ is open for every $\alpha \in \mathcal{A}$. Again by definition of subset topology, we have that $f^{-1}(W_\alpha) \cap B$ is an open subset of B for every α. Because $\bigcup_{\alpha \in \mathcal{A}} f^{-1}(W_\alpha) \cap B = B$ (I will leave the verification to you), we conclude that

$$\{f^{-1}(W_\alpha) \cap B\}_{\alpha \in \mathcal{A}}$$

is an open cover of B. Finally, we use the compactness of B: Because B is compact, there is some finite subset $\{\alpha_1, \ldots, \alpha_n\} \subset \mathcal{A}$ so that

$$f^{-1}(V_{\alpha_1}) \cup \ldots \cup f^{-1}(V_{\alpha_n}) = B.$$

Thus,

$$V_{\alpha_1} \cup \ldots \cup V_{\alpha_n} = f(B).$$

This means we have exhibited a finite subcover of $\{V_\alpha\}_{\alpha \in \mathcal{A}}$. In sum, for any open cover of $f(A)$, we were able to exhibit a finite subcover. This means $f(A)$ is compact.