Homework 9 Solutions

9.1 Proof (10 points)

Suppose X is compact and Y is Hausdorff.

1. Show that if $A \subset X$ is closed, then A is compact.

2. Show that if $B \subset Y$ is compact, then it is closed.

3. Let $f : X \to Y$ be a continuous bijection. Then show f is a homeomorphism (still under the assumption that X is compact and Y is Hausdorff). (Hint: Use the previous parts of this problem, and use last week's proof, too.)

(In what follows, I am using $\mathcal{D}$ to index the elements of an open cover; this is to avoid the confusion between the usual indexing set $\mathcal{A}$ and the subspace A.)

1. Let $\{V_\delta\}_{\delta \in \mathcal{D}}$ be an open cover of A. By definition of subset topology, this means that for every $\delta \in \mathcal{C}$, there exists $W_\delta \subset X$ such that W_δ is open in X, and $V_\delta = W_\delta \cap A$. Now, because $A \subset X$ is closed, the complement A^C is open. We conclude that the collection

$$\mathcal{U} = \{A^C\} \bigcup \{W_\delta\}_{\delta \in \mathcal{D}}$$

is an open cover of X. Because X is compact, $\mathcal{U}$ admits a finite subcover; let us denote by

$$W_{\delta_1}, \ldots, W_{\delta_n}$$

the finite collection of W_δ in this finite subcover. (That is, we ignore whether or not A^C shows up in the finite subcover.) Then it is straightforward to check that the collection

$$\{W_{\delta_1} \cap A, \ldots, W_{\delta_n} \cap A\}$$

is a cover of A. Because (by definition of the W_δ) $V_{\delta_i} = W_{\delta_i} \cap A$, we conclude that

$$\{V_{\delta_1}, \ldots, V_{\delta_n}\}$$

is a finite subcover of the original open cover. This shows A is compact.

2. Let $B \subset Y$ be compact. We must show that the complement $B^C \subset Y$ is open. To show that B^C is open, it suffices to show that for every $y' \in B^C$, there exists some open subset $V' \subset Y$ such that $V' \subset B^C$ and $y' \in V'$. (For then we can write B^C as the union of the V', exhibiting B^C as open.)

So fix $y' \in B^C$ and let us exhibit an open set $V' \subset Y$ for which $y' \in V' \subset B^C$. For every $y \in B$, the Hausdorffness of Y guarantees the existence of two open sets

$$U_{y,y'} \subset Y, \qquad U'_{y,y'} \subset Y$$

such that $U_{y,y'} \cap U'_{y,y'} = \emptyset$ and $y \in U_{y,y'}$, $y' \in U'_{y,y'}$. By definition of subspace topology, each $U_{y,y'} \cap B$ is an open subset of B. Hence the collection

$$\{U_{y,y'} \cap B\}_{(y,y')} \in B \times B^C\}$$

is an open cover of B. Because B is compact, there is a finite subcover—that is, a finite collection $\{(y_i, y')\}_{i=1,\ldots,n}$ so that

$$\bigcup_{i=1,\ldots,n} (U_{y_i,y'} \cap B) = B.$$

Now let us define the following intersection:

$$V' := U'_{y_1,y'} \cap \ldots \cap U'_{y_n,y'}.$$

By de Morgan's laws, it is straightforward to check that V' does not intersect $U_{y_1,y'}, \ldots, U_{y_n,y'}$; and in particular, V' does not intersect B. This means that $V' \subset B^C$. Because V' is defined as a *finite* intersection of open subsets of Y, it is also open. We are finished.

3. Because $f : X \to Y$ is already a continuous bijection, we must prove that the inverse f^{-1} is continuous. From class, we know that f^{-1} is continuous if and only if for every closed $A \subset X$, we have that the preimage of A under f^{-1} is closed in Y. So, we must show that if $A \subset X$ is closed, then $f(A) \subset Y$ is closed. Well, from part 1 of this problem, we know that if $A \subset X$ is closed, then A is compact. From last week's homework, we know that $f(A)$ is also compact. By part 2 of this problem, we know that $f(A) \subset Y$ is closed. This finishes the proof.