# Homework 10 Solutions

## 10.1 Proof (10 points)

(a) Prove that the quotient of any compact space is compact. That is, if $X$ is a compact topological space, and $\sim$ an equivalence relation on $X$, then $X/\sim$ is compact (with the quotient topology).

(b) Prove that $\mathbb{R}P^2$ is compact.

(c) Look back on the homework assignment requiring you to prove that $S^1/\sim$ is homeomorphic to $\mathbb{R}^2 \setminus \{(0,0)\}/\sim$. Can some part of that proof be simplified in light of facts you've learned this week and last?

(a) We know from previous results that the quotient map $p : X \to X/\sim$ is continuous; thus, using a result from a previous homework, we see that if $A \subset X$ is any compact subspace, $p(A)$ is compact. Thus $p(X)$ is compact. On the other hand, $p$ is also a surjection by definition of the quotient map; hence $p(A) = X/\sim$ is compact.

(b) We know that $\mathbb{R}P^2 = S^2/\sim$, where $\sim$ is the relation saying $x \sim \pm x$ for any $x \in S^2$. On the other hand, $S^2 \subset \mathbb{R}^3$ is compact by the Heine-Borel theorem (because $S^1$ is both closed and bounded—in case this isn't clear, note that $S^2$ is closed because it is the solution to a polynomial equation: $x_1^2 + x_2^2 + x_3^2 - 1 = 0$.) Thus by part (a), we see that $\mathbb{R}P^2$ is compact.

(c) Yes; we needed to show that a map $f : S^1/\sim \to (\mathbb{R}^2 \setminus \{0\})/\sim$ was a homeomorphism in a previous homework. It is not hard to show that the target is Hausdorff—for any two points $[x]$ and $[y]$ that are not equal, choose open sectors $U, V \subset \mathbb{R}^2 \setminus \{0\}$ such that $U \cap V = \emptyset$ but $x \in U$ and $y \in V$. Knowing that the domain fo $f$ is now compact, and invoking a previous homework assignment, all we need to show is that $f$ is continuous and a bijection, without worrying about whether the inverse is continuous.