Homework 11 Solutions

Proof (10 points)

Let (X, d) be a metric space, and fix a sequence of elements

$$x_1, x_2, \ldots$$

in X. Choose also $x \in X$.

Recall that we say this sequence *converges to* x if, for every $\epsilon > 0$, there exists $N > 0$ so that

$$i > N \implies d(x_i, x) < \epsilon.$$

(a) Suppose X is a metric space and that $A \subset X$ is a closed subset. Then for any convergent sequence $x_1, x_2, \ldots$ converging to x, if $x_i \in A$ for all i, then $x \in A$.

(b) Suppose X is compact, and let $f : X \to \mathbb{R}$ be a continuous function. Show that f achieves a minimum and a maximum—that is, there is some $x_{\min} \in X$ such that for all $x \in X$, $f(x) \geq f(x_{\min})$. Likewise, there is some $x_{\max} \in X$ so that for all $x \in X$, $f(x) \leq f(x_{\max})$.

(a) Note that A^C is open because A is closed. Thus if $x \in A^C$ we know (by definition of topology induced by a metric) that there is some $\epsilon > 0$ for which $\text{Ball}(x; \epsilon) \subset A^C$. In particular, there is no point in A which is within distance ϵ of x. Thus no sequence in X could converge to $x \in A^C$ while being fully contained in A. This proves the contrapositive, hence the original statement.

(b) We only prove the statement about the maximal element; the proof of the minimal element is similar, or can be proven by examining the function $-f$, which is a composition of f with the continuous function $t \mapsto -t$.
Because X is compact and f is continuous, the image $f(X)$ is compact. By the Heine-Borel theorem, this means that $f(X)$ is a closed and bounded subset of $\mathbb{R}$. Because $f(X)$ is bounded, let us choose t to be the least upper bound of $f(X)$, otherwise known as the supremum of $f(X)$. Then because $f(X)$ is closed, we know that $t \in f(X)$, hence there exists some $x \in X$ for which $f(x) = t$, and hence $f(x) \geq f(x')$ for any $x' \in X$.
(Here is a more involved argument: Let M be the set of all $t \in \mathbb{R}$ such that $x \in X \implies t \geq f(x)$. Choose any $x_0 \in X$, and any $t_0 \in M$. Now inductively define x_{n+1} and t_{n+1} as follows: If the average $s = (t_n - x_n)/2$ is in M, let t_{n+1} be this average and let $x_{n+1} = x_n$; otherwise, choose x_{n+1} to be any element of X such that $f(x_{n+1})$ is a number between s and t_n, and set t_{n+1} to equal t_n. Then the sequence $f(x_0), f(x_1), \ldots$ forms an increasing, bounded sequence, and hence have a limit in $\mathbb{R}$. By a previous homework problem, because $f(X)$ is closed, this limit is in $f(X)$ itself; thus there is some $x \in X$ for which $f(x)$ is the limit. By construction, $f(x)$ satisfies the property that it is smaller than or equal to any element of M, and larger than or equal to any element in $f(X)$. This finishes the proof.)