Homework 12 Solutions

Proof (10 points)

Let $n = (0,0,1) \in \mathbb{R}^3$ be the north pole of the sphere. Let

$$p : S^2 \setminus \{n\} \to \mathbb{R}^2, \qquad (x_1, x_2, x_3) \mapsto \frac{1}{1 - x_3}(x_1, x_2)$$

be the stereographic projection. Show that the function

$$S^2 \to (\mathbb{R}^2)^+ = \mathbb{R}^2 \cup \{*\}, \qquad x \mapsto \begin{cases} * & x = n \\ p(x) & x \neq n \end{cases}$$

(the codomain is the one-point compactification of $\mathbb{R}^2$) is a homeomorphism.

Let's set some notation. We'll call the function f. We'll denote elements of the domain by x, and elements of the codomain by y.
We first show f is a surjection. Clearly the point $* \in (\mathbb{R}^2)^+$ is equal to $f(n)$, so we need only check that all points of $\mathbb{R}^2$ are in the image of f. That is, given $y = (y_1, y_2)$, we seek (x_1, x_2, x_3) such that

$$x_1^2 + x_2^2 + x_3^2 = 1 \qquad \text{and} \qquad y_1 = \frac{x_1}{1 - x_3}, \qquad y_1 = \frac{x_2}{1 - x_3} \tag{12.1}$$

Note we are finished if we can solve for x_3, for then our knowledge of each y_i tells us that $x_i = (1 - x_3)y_i$. The equation of the sphere tells us

$$\begin{aligned} 0 &= ((1 - x_3)y_1)^2 + ((1 - x_3)y_2)^2 + x_3^2 - 1 \\ &= (1 + y_1^2 + y_2^2)x_3^2 + -2(y_1^2 + y_2^2)x_3 + (y_1^2 + y_2^2 - 1). \end{aligned}$$

Thus we can solve for x_3 using the quadratic formula:

$$\begin{aligned} x_3 &= \frac{2(y_1^2 + y_2^2) \pm \sqrt{4(y_1^2 + y_2^2)^2 - 4(1 + y_1^2 + y_2^2)(-1 + y_1^2 + y_2^2)}}{2(1 + y_1^2 + y_2^2)} \\ &= \frac{y_1^2 + y_2^2 \pm \sqrt{(y_1^2 + y_2^2)^2 - (y_1^2 + y_2^2)^2 + 1}}{1 + y_1^2 + y_2^2} \\ &= \frac{\pm 1 + y_1^2 + y_2^2}{1 + y_1^2 + y_2^2} \end{aligned} \tag{12.2}$$

The solution $x_3 = 1$ clearly falls outside our scope, as such a point is not in $S^2 \setminus \{n\}$. Thus we find

$$x_3 = \frac{y_1^2 + y_2^2 - 1}{y_1^2 + y_2^2 + 1}, \qquad x_1 = y_1(1 - x_3), \qquad x_2 = y_2(1 - x_3) \tag{12.3}$$

Incidentally, this also shows that f is a bijection—this is because we found that any x satisfying (12.1) must have coordinates given by (12.3).

Now we must show that f is continuous. First, let us show that the map

$$\phi : S^2 \setminus \{n\} \to \mathbb{R}^2, \qquad x \mapsto f(x)$$

is continuous. To see this, we first note that each of the following functions is continuous:

$$h : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{0\}, \qquad h(t) = 1 - t, \qquad g : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \qquad g(s) = 1/s.$$

Thus, the composition

$$\mathbb{R}\times\mathbb{R}\times(\mathbb{R}\setminus\{1\}) \to \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \qquad (x_1, x_2, x_3) \mapsto (gh(x_3), x_1) \mapsto gh(x_3)\cdot x_1 = \frac{x_1}{1 - x_2}$$

is continuous. Likewise for the map $(x_1, x_2, x_3) \mapsto \frac{x_2}{1-x_3}$. Hence the composition

$$S^2 \setminus \{n\} \to \mathbb{R} \times \mathbb{R} \times (\mathbb{R} \setminus \{1\}) \to \mathbb{R} \times \mathbb{R}, \qquad (x_1, x_2, x_3) \mapsto \frac{1}{1 - x_3}(x_1, x_2)$$

is continuous. But this composition is ϕ!
And we learn more: We know ϕ is a bijection because ϕ^{-1} is given by (12.3). By similar reasoning as above, we see that ϕ^{-1} is continuous.
Now suppose $U \subset (\mathbb{R}^2)^+$ does not intersect $*$. Then $f^{-1}(U) = \phi^{-1}(U)$, which—by the continuity of ϕ—is an open subset of $S^2 \setminus \{n\}$. But S^2 is a metric space, hence Hausdorff, so $\{n\}$ is a closed subset of S^2, meaning $S^2 \setminus \{n\}$ is open in S^2. In particular, $f^{-1}(U) = \phi^{-1}(U) = \phi^{-1}(U) \cap S^2 \setminus \{n\}$ is open.
Now suppose that U does intersect $*$. By definition of the one-point compactification, this means that $U \cap \mathbb{R}^2 = \mathbb{R}^2 \setminus K$, where K is some compact subset of $\mathbb{R}^2$. Thus $f^{-1}(U) = f^{-1}(K)^C = \phi^{-1}(K)^C$. Since ϕ^{-1} is continuous, we know that $\phi^{-1}(K)$ is also compact; meaning by Heine-Borel that it is a closed (and bounded) subset of $\mathbb{R}^3$, so that in particular $\phi^{-1}(K) \cap S^2$ is also closed. This means that $f^{-1}(U)^C$ is closed in S^2, so that $f^{-1}(U)$ is open in S^2.
Thus, for all open $U \subset (\mathbb{R}^2)^+$, we have shown that $f^{-1}(U)$ is open in S^2. This shows that f is continuous.

Now we conclude by showing that $(\mathbb{R}^2)^+$ is Hausdorff. Fix $y, y' \in (\mathbb{R}^2)^+$. If both y, y' are in $\mathbb{R}^2$, this is obvious, as $\mathbb{R}^2$ is Hausdorff, and any open subset of $\mathbb{R}^2$ is open in its one-point compactification by definition. Now suppose that $y' = *$. Let $U = \text{Ball}(0, 2|y|)$ be an open ball of radius $2|y|$ (or any radius large enough so that y is in the open ball). Let K be the closed ball of same radius; K is closed and bounded, so compact by Heine-Borel. We let $U' = (\mathbb{R}^2)^+ \setminus K$, so that U' is open by definition of one-point compactification. Clearly $* \in U'$, and by design, $U \cap U' = \emptyset$. This shows that $(\mathbb{R}^2)^+$ is Hausdorff.
Thus, because the domain of f is compact, the codomain is Hausdorff, and f is a continuous bijection, we have shown that f is a homeomorphism.