Introduction

Welcome to class. Here are some details:

1. Me: Hiro
2. You: Taking Math 4330, General topology
3. My e-mail, office hours, office, handing out the syllabus, et cetera.

Now let's get to the good stuff.

Topology is the study of shapes.

Question 0.0.0.1. At this point, what questions do you have as a student?

(Here, I field questions. But I move forward with two:)

1. What do you mean by shape?
2. How do you study them?

The purpose of this class is to give you the vocabulary to begin understanding the answers to these questions.

Remark 0.0.0.2. But the vocabulary of mathematics is not like vocabulary of foreign language; these will not be new words for old ideas; these will be new words for new ideas.

Remark 0.0.0.3. Just as it will all take us many years to learn what love is, just as we will have to update our understanding as time passes, and just as this conceptualization will only change fruitfully as you invest time in this idea of love, your idea of the word "space" will also require both the passage and investment of time to develop. Be patient with yourself.

0.1 A moment of confusion; our goals

Let me open the floodgates for a moment to lay on you some definitions. You may see terms you are not familiar with, and what happens in the next five minutes, you are not responsible for knowing just yet.

Definition 0.1.0.1. A *topological space* is the data of a pair

$$(X, \mathcal{T})$$

where X is a set, and $\mathcal{T}$ is a collection of subsets of X, satisfying the following conditions:

1. The empty set $\emptyset$ and X itself are in $\mathcal{T}$,
2. For any finite collection $U_1, \ldots, U_n$ in $\mathcal{T}$, the intersection $U_1 \cap \ldots \cap U_n$ is in $\mathcal{T}$, and
3. For *any* collection $\{U_\alpha\} \subset \mathcal{T}$, the union $\bigcup_\alpha U_\alpha$ is in $\mathcal{T}$.

Definition 0.1.0.2. Let $(X, \mathcal{T})$ be a topological space. An element $U \in \mathcal{T}$ is called an *open set* of X.

Definition 0.1.0.3. Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be two topological spaces. A function $f : X \to X'$ is called *continuous* if for any open set $U' \in \mathcal{T}'$, the preimage $f^{-1}(U')$ is an open set of X.

That took a few minutes. Believe it or not, if you understand the above three definitions, you have completed at least half the class.

But you do not understand, at least at this moment. A major *goal* of this class will be to understand the above definitions, and as I said, this will take time. We have the semester for a reason.

Another major goal of this class will be for you to begin thinking the way mathematicians do. This means to understand how to *come* to an understanding.

0.2 Continuity

So let's get to it.

Here's an often unspoken tip about being a mathematician: Oftentimes, we give definitions of objects, only to be able to understand the *functions* between them.

Example 0.2.0.1. For now, you can pretend that I only gave the definition of a topological space so that I can tell you what a continuous function is.

At this point I want you to feel funny: You already know what a continuous function is! (Or you're supposed to, at least.) Let's review.

Let us fix a function

$$f : \mathbb{R} \to \mathbb{R}.$$

I will take some time to dissect this notation for you.

1. Here, $\mathbb{R}$ is the collection of all real numbers. It is a set. It contains things like 0, 1, 2, 3, π, $-\pi$, e, $3/4$, $\sqrt{2}$, and so forth.

2. The colon :, along with the arrow $\to$, indicates that I am defining a function. This function has domain $\mathbb{R}$, and target $\mathbb{R}$ as well. In plain English, this means I am defining an assignment which eats a real number, and spits out a (possibly different) real number. An example would be something that takes a real number and outputs its square; this is often referred to as the function $f(x) = x^2$.

3. The letter f indicates the *name* I want to give to the function. For example, if I were to write "$g : \mathbb{R} \to \mathbb{R}$," I am merely declaring that from hereon, I will be talking about a function called g.

4. I used the phrase "let us fix a function f." This is jargon, the same way lawyers use legal terms, mathematicians use their own linguistic conventions. "Let us fix a function" does not mean that we all choose our favorite function. "Let us fix a function," in fact, means almost the opposite—it means that we are about to discuss something that is true for an *arbitrary* function. You are allowed to have a function in mind, but you must also be aware that the devil may be in the room, and the devil may choose a completely different (and horrible-looking) function.

Discussion 0.2.0.2. What does it mean for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous?

(Some discussions will talk about intuitive meanings, which is fine. Some discussions may be imprecise. Some will get at a definition.)

In this discussion, I expect some ideas to come up. Things like:

1. The graph of f has no “jumps.”
2. The graph of f is “connected.”
3. The graph of f “divides” the plane into two halves.
4. If a sequence x_n converges to x, then the sequence $f(x_n)$ converges to $f(x)$.
5. f satisfies the “epsilon-delta” definition.

My expectation is that almost everybody will have some intuition—some correct, some incorrect–about what a continuous function is. I suspect only a few people will have remembered what the definition that you learn in calculus is:

Definition 0.2.0.3. A function $f : \mathbb{R} \to \mathbb{R}$ is called *continuous* if and only if:

For every x and for every $\epsilon > 0$,
there exists a $\delta > 0$ so that
for every x', we have

$$|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon.$$

At this point, you have seen a “definition” of continuity (Definition 0.2.0.3) and you have also discussed your *intuition* of continuous functions. You should notice that the definition and the intuition may look very different.

So, we have already seen one of the major ideas of modern mathematics, and of this class: Continuity. We will talk more about this as time goes on, of course. For next time, all you need to do is explore this idea on your own terms, and turn in the result.

0.3 Getting to know $\mathbb{R}$

As my preview might have hinted, we need to understand and define what it means for a function to be continuous even when the domain and codomain may not be $\mathbb{R}$. To do that, we will now examine what enabled us to define a notion of continuity for functions from $\mathbb{R}$ to $\mathbb{R}$; understanding the ingredients in our familiar case will allow us to extend our ideas to the unfamiliar cases.

Remark 0.3.0.1. You have known $\mathbb{R}$—the set of real numbers—for a long time. But like a family member you have known a long time, sometimes it is only with intense reflection that you realize the things you have taken for granted.

You are, believe it or not, very familiar with $\mathbb{R}$—like family. But what are we relying on to define continuity?

Discussion 0.3.0.2. What *properties* or *structures* of $\mathbb{R}$ are we using in the epsilon-delta definition of continuity (Definition 0.2.0.3)?

Some things you may come up with:

1. We know how to "subtract" elements when we write things like $x - x'$ or $f(x) - f(x')$.
2. We know how to take absolute value when we write something like $|x - x'|$ (or $|f(x) - f(x')|$.
3. We know how to *compare* $|x - x'|$ with ϵ so we can write something like ϵ.
4. In fact, we also have intuition about the first two things Hiro listed: "subtracting" and "take absolute value" combine to give us a notion of "distance" between two points—$|x - x'|$ is the distance from x to x'.

I would now like to focus on this idea of *distance.* This will lead us to one of the most intuitive ways to talk about spaces and continuous maps between them.

Metric spaces

As I've mentioned before, we will be very much interested in notions of distance. This is because—at least based on our everyday experiences—whenever we think of a shape or a space, we can certainly measure the distance between two points on that shape or space. Moreover, we saw in the previous section that the very definition of continuity (for functions $f : \mathbb{R} \to \mathbb{R}$) utilized the notion of distance.

Remark 0.3.0.3. Later in our course, we will study shapes where it is *unnatural* to speak of distances; this may come as a surprise, but more on that later.

Today, we'll isolate what properties of "distance" are reasonable to expect on the kinds of shapes we're familiar with.

0.4 Preliminaries: Products

First, let me remind you of some background; it's okay if this is the first time you've seen these background ideas.

Definition 0.4.0.1. Let X and Y be two sets. Then the notation

$$X \times Y$$

represents their product; this is also sometimes called the *Cartesian product* of X and Y.

$X \times Y$ is a set whose elements are ordered pairs

$$(x, y)$$

with $x \in X$ and $y \in Y$.

Example 0.4.0.2. Let X be a set of three people named Alejandra, Bill, and Candace. Let Y be a set of two people named Seungwan and Theo. Then $X \times Y$ has exactly six elements, and they are listed as follows:

- (Alejandra, Seungwan)
- (Bill, Seungwan)
- (Candace, Seungwan)
- (Alejandra, Theo)
- (Bill, Theo)
- (Candace, Theo)

Note that (Theo, Candace) is *not* an element of $X \times Y$. This is what the word "ordered" means in "ordered pair."

Example 0.4.0.3. X and Y may be the same set. For example, let $\mathbb{R}$ be the set of all real numbers, and set $X = Y = \mathbb{R}$. Then $X \times Y$ has another names, called $\mathbb{R}^2$.[1]

We will often denote an element of $\mathbb{R}^2$ by (x_1, x_2).

Example 0.4.0.4 (Iterated products)**.** You can iterate the product construction. For example, if you have three sets X and Y and Z, it makes sense to form the sets

$$(X \times Y) \times Z \qquad \text{and} \qquad X \times (Y \times Z).$$

These two sets are *not* the same, but there is a natural bijection between them. This distinction need not worry you for the time being, but thinking through this statement carefully will do you a lot of good in the future.

There is yet another set you can construct, which we will write

$$X \times Y \times Z.$$

The elements of $X \times Y \times Z$ consist of ordered triplets (x, y, z) where $x \in X, y \in Y$ and $z \in Z$.

Of course if you have a collection of sets, you can take the product of all of them.

[1] This explains the notation $\mathbb{R}^2$; it is quite informal and lazy, but the rationale behind the notation is the suggestive equality $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Example 0.4.0.5 (Euclidean space)**.** An important example is $\mathbb{R}^n$, which is the n-fold Cartesian product of $\mathbb{R}$. You may be more familiar thinking of $\mathbb{R}^n$ as n-dimensional Euclidean space.

Example 0.4.0.6. Fix a set X. We will soon think about functions

$$X \times X \to \mathbb{R}.$$

This means that, for every ordered pair of elements (x_1, x_2) with $x_1, x_2 \in X$, we will assign a real number.

When $X = \mathbb{R}$, you have seen many examples of such functions:

1. Addition, which sends a pair (x_1, x_2) to $x_1 + x_2$.

2. Subtraction, which sends (x_1, x_2) to $x_1 - x_2$.

3. Multiplication, which sends a pair (x_1, x_2) to the product $x_1 \cdot x_2$.

4. Division is *not* an example of a function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. While you may happily write a formula taking the pair (x_1, x_2) to the quotient x_1/x_2, this is not defined when $x_2 = 0$.

5. The distance function, which takes a pair (x_1, x_2) to the distance between them: $|x_2 - x_1|$.

0.5 Definition of metric spaces

Notation 0.5.0.1. Let X be a set. We will often write an element of $X \times X$ as (x, x'). (In the previous section, we used the notation (x_1, x_2) instead.) The symbol x' is read "x prime." The reason for this is that we will soon let $X = \mathbb{R}^n$, so that X itself is made up of ordered tuples; the dual roles of subscripts will then become quite confusing, so we will use the "prime" symbol.

Example 0.5.0.2 (Distance on $\mathbb{R}^2$)**.** We've already talked about the distance function on the set $X = \mathbb{R}$:

$$d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x, x') \mapsto |x' - x|.$$

Let's now think about $X = \mathbb{R}^2$. Given two points in $\mathbb{R}^2$, what is the distance between them?

The Pythagorean theorem tells us: Given two points $x = (x_1, x_2)$ and $x' = (x'_1, x'_2)$ in $\mathbb{R}^2$, the length of the path between them is given by

$$d(x, x') = \sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2}. \tag{0.5.1}$$

Note that this function has a lot of intuitive properties:

1. If $x = x'$, then the distance between x and x' is zero.
2. Conversely, if the distance between two points is zero, they are equal points.
3. The triangle inequality: This is not always intuitive for most students, but it is a fact of life. If you have three points x, x', x'', then the distance from x to x'' is *at most* the sum of the distances between x and x', and between x' and x''. (You should draw a picture.)
4. Symmetry: The distance from x to x' is the same as the distance from x' to x.

There are others, but we will leave that for exercises or personal exploration.

Example 0.5.0.3. Now let's consider a different shape X. For example, let's take X to be any *arbitrary* subset of $\mathbb{R}^2$.

Is there still a notion of distance between two points of X? Yes; you could just measure the distance as you normally would inside $\mathbb{R}^2$. Thus we have a function

$$d : X \times X \to \mathbb{R}$$

by the exact same formula in (0.5.1).

Does this function satisfy all the properties we talked about in Example 0.5.0.2? Yes.

We isolate these properties to give the following definition:

Definition 0.5.0.4. A *metric space* is the data of a pair (X, d) where X is a set, and

$$d : X \times X \to \mathbb{R}$$

is a function satisfying the following properties:

(0) $d(x, x') = 0 \iff x = x'$.

(1) (Symmetry) $d(x, x') = d(x', x)$.

(2) (Triangle inequality) $d(x, x') + d(x', x'') \geq d(x, x'')$.

Remark 0.5.0.5. Intuitively, a metric space is a set with some notion of distance between two points. Note that a single set X may admit many different examples of a function d. When should we consider to metric spaces to be equivalent? We will get to that in Section 0.8.

Exercise 0.5.0.6. Show that if (X, d) is a metric space, then for any pair $x, x' \in X$, we have that $d(x, x') \geq 0$.

At this point, what questions do you have?

0.6 Continuous maps

Definition 0.6.0.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Fix a function $f : X \to Y$. We say that f is *continuous* if:

For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')).$$

Remark 0.6.0.2. Informally, the above definition says that a continuous function between metric spaces is one that respects the idea of *closeness*.

You should think of ϵ as a number a mortal enemy gives you, daring you to be ϵ-close (i.e., within ϵ) of $f(x)$.

You should think of δ as the number that allows you to vanquish that dare: If x' is any element δ-close to x, then you know that $f(x')$ is ϵ-close to $f(x)$.

0.7 Examples of metric spaces

These are all useful examples. You should do your best to understand them.

Example 0.7.0.1 (Euclidean space)**.** Let $X = \mathbb{R}^n$. Define

$$d(x, x') = \sqrt{\sum_{i=1}^{n} (x'_i - x_i)^2}.$$

This is called the "standard" metric on $\mathbb{R}^n$.