Last time:

Defn A fxn $f: \mathbb{R} \longrightarrow \mathbb{R}$ is called continuous, or C^0, if

$$\forall\, x \in \mathbb{R},\ \forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ s.t.}$$

$$d(x', x) < \delta \implies d(f(x'), f(x)) < \varepsilon.$$

Defn A metric space is a set X together w/ a function

$$d: X \times X \longrightarrow \mathbb{R}$$

such that

(0) $d(x, x') = 0 \iff x = x'$

(1) $d(x, x') = d(x', x)$

(2) $d(x, x') + d(x', x'') \geq d(x, x'')$.

<u>Defn</u> Let (X, d_X) and (Y, d_Y) be metric spaces. Then a function
$$f : X \longrightarrow Y$$
is called <u>continuous</u>, or C^0, if

$$\forall x \in X, \quad \forall \varepsilon > 0, \quad \exists \delta > 0 \text{ s.t.}$$

$$d_X(x', x) < \delta \Rightarrow d_Y(f(x'), f(x)) < \varepsilon.$$

<u>Rmk</u> I motivated the idea of a metric space by asking: what do we need to <u>define</u> continuity? The ε-δ definition suggested all we need is a notion of <u>distance</u>. That's what the function d captures.

Intuition for epsilon-delta definition: "Continuity" for a map between metric spaces means that the map "respects closeness" in the following sense: For any epsilon, you can guarantee that you'll end up epsilon-close so long as you start out delta-close.

At this point, what should you demand of me?

(As with anybody else who gives you a new definition.)

- Examples
- Motivation (i.e., why is this useful?)

Before we move on, let me mention:

<u>Defn</u> Let (X,d) be a metric space. Then d is called a <u>metric</u> on X.

The standard metric

Example Let $X=\mathbb{R}$, and

$$d: \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$$
$$(x,x') \longmapsto |x'-x|.$$

Exer Verify this example is indeed a metric space.

Example Let $X=\mathbb{R}^n$, and

$$d: X\times X \longrightarrow \mathbb{R}$$
$$(x,x') \longmapsto \sqrt{\sum_{i=1}^{n} (x_i'-x_i)^2}.$$

Exer Show this is a metric space.

Solution: Check all three conditions:

($\mathbb{R}$) (0) If $d(x,x')=0$, then $|x'-x|=0$.

- i.e., $x'-x=0$. So $x=x'$.

If $x=x'$, then $d(x,x')=|x-x|=|0|=0$.

(1) $d(x,x') = |x'-x|$

$= |x-x'|$ (b/c $|x-x'| = |-(x'-x)| = |x'-x|$)

$= d(x',x)$

(2) $d(x,x')+d(x',x'') = |x'-x| + |x''-x'|$

$\geq |(x'-x)+(x''-x')|$ $\star$

$= |x''-x'+x'-x|$

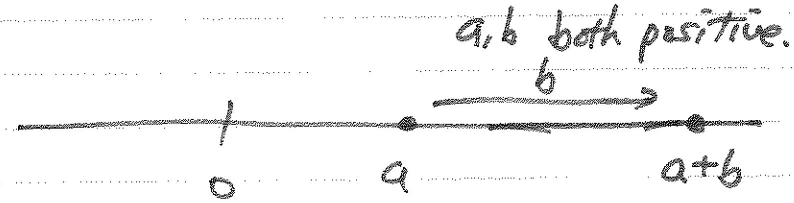
$= |x''-x|$

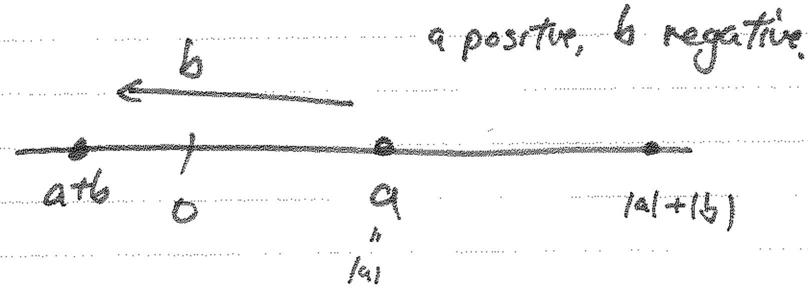
$= d(x,x'')$.

__Rmk__ Make sure you understand $(\ast)$. For any pair of real numbers (a,b), we have

$$|a+b| \leq |a| + |b|.$$

(If a and b have same sign, then $|a+b| = |a| + |b|$. Otherwise, $|a+b| < |a| + |b|$.)





We'll verify that $\sqrt{\sum (x_i' - x_i)^2}$ is a metric another time; for now let's see other examples.

The discrete metric

One of the more suprising facts is that $\mathbb{R}^n$ has many diffent metric space structures you can put on it.

Ex (The discrete metric space structure).

Define

$$d(x,x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x' . \end{cases} \qquad (\ast\ast)$$

Exer Show this makes $\mathbb{R}^n$ into a metric space. What did you use about $\mathbb{R}^n$?

Sol'n:

(0) $x = x' \Longrightarrow d(x, x') = 0$ by def'n.

$d(x, x') = 0 \Longrightarrow d(x, x') \neq 1$

$\Longrightarrow x = x'$ by def'n.

(1) If $x = x'$, $d(x, x') = 0 = d(x', x)$.

If $x \neq x'$, $d(x, x') = 1 = d(x', x)$.

(2) $$d(x, x') + d(x', x'') = \begin{cases} 1 + 1 & x \neq x' \text{ and } x' \neq x'' \\ 1 + 0 & \\ 0 + 1 & \\ 0 + 0 & x = x' = x'' \end{cases}$$

while

$$d(x, x'') = \begin{cases} 1 & x \neq x'' \\ 0 & x = x'' \end{cases}.$$

//

We used <u>nothing</u> about $\mathbb{R}^n$.

That is, we observe:

Ex Let X be any set. Then $(\star\star)$ defines a metric space structure on X.

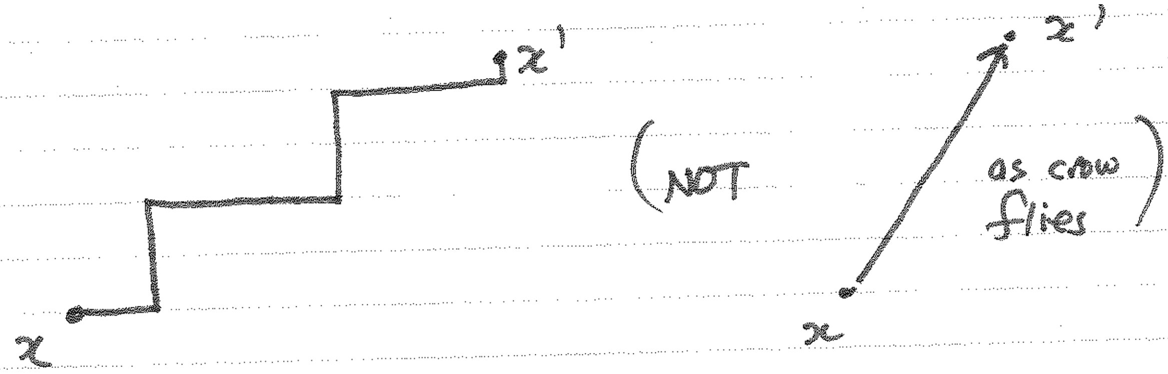
Rmk In fact, the number 1 is irrelevant — for any number $C > 0$, the assignment

$$d(x, x') = \begin{cases} 0 & x = x' \\ C & x \neq x' \end{cases}$$

is a metric.

Manhattan metric

If you lived in a city w/ a grid system (e.g., Manhattan), you wouldn't be able to get from x to x' as the crow flies; you'd need to walk along blocks:



what's the distance you've traveled if you walk along the grid?

$$|x_1' - x_1| + |x_2' - x_2|.$$

Exer Show

$$d(x,x') = \sum_{i=1}^{n} |x_i' - x_i|$$

is a metric on $\mathbb{R}^n$.

Sol'n: (0) $x = x' \Longrightarrow |x_i' - x_i| = |0| = 0$ $\forall$ i, so

$$\Longrightarrow \sum_{i=1}^{n} |x_i' - x_i|$$

$$= \sum_{i=1}^{n} 0$$

$$= 0.$$

On the other hand, if $d(x, x') = 0$, then (because $|\cdot| \geq 0$) it must be that $\forall i$, $|x_i' - x_i| = 0$. Hence $x = x'$.

(1) $|x_i' - x_i| = |-(x_i' - x_i)| = |x_i - x_i'|$ so $d(x', x) = d(x, x')$.

(2) $d(x, x') + d(x', x'') = \sum |x_i' - x_i| + \sum |x_i'' - x_i'|$.

For all i, we have:

$$|x_i' - x_i| + |x_i'' - x_i'| \geq |x_i'' - x_i' + (x_i' - x_i)|$$

$$= |x_i'' - x_i|$$

Hence

$$d(x,x')+d(x',x'') \geq \sum_{i=1}^{n} |x_i''-x_i|$$

$$= d(x,x''). \quad //$$

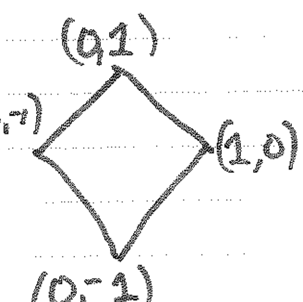
<u>Rmk</u> The Manhattan metric is sometimes called the taxicab metric.

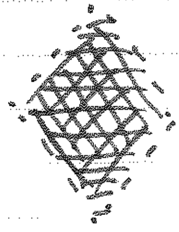
<u>Exer</u> Draw the following subsets of $\mathbb{R}^2$:

(a) $\{x \text{ s.t. } d(x,0)=1\}$

(b) $\{x \text{ s.t. } d(x,0)<1\}$.

(using the taxicab metric).

<u>Sol'n</u> (a) 

(b) 

The ℓ^∞ metric

(pronounced "ell infinity")

Let

$$d(x, x') = \max_{i=1,\cdots,n} |x_i' - x_i|.$$

That is, we declare the distance from x to x' to be the largest of the distance between their coordinates.

Ex $x = (2, 2, 2)$ $\quad x' = (3, 4, 5)$

then $|x_1' - x_1| = 1 \quad (3-2)$

$|x_2' - x_2| = |4-2| = 2$

$|x_3' - x_3| = |5-2| = 3.$

$$\max_i |x_i' - x_i| = \max\{1, 2, 3\}$$
$$= 3.$$

So $d(x', x) = 3.$

Exer Show this is a metric on $\mathbb{R}^n$.

Soln: (0) $x' = x \Rightarrow \forall i,\ |x_i' - x_i| = 0$

$$\Rightarrow \max_i |x_i' - x_i| = 0.$$

If $\max_i |x_i' - x_i| = 0$, then $|x_i' - x_i| = 0$ for all i.

Hence $x_i = x_i'\ \forall i$, hence $x' = x$.

(1) $d(x', x) = d(x, x')$ is straightforward.

(2) For all i, we have

$$|x_i'' - x_i| \le |x_i'' - x_i'| + |x_i' - x_i|$$

$$\le \max_{j=1,\dots,n} |x_j'' - x_j'| + \max_{k=1,\dots,n} |x_k' - x_k|$$

$$= d(x'', x') + d(x, x').$$

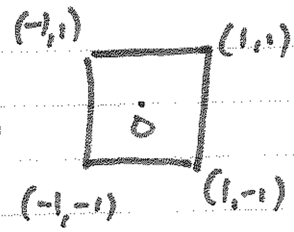
Hence $d(x, x'') = \max_i |x_i'' - x_i| \le d(x'', x') + d(x, x')$. //

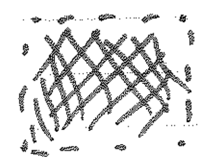
Exer Draw

(a) $\{x \text{ s.t. } d(x,0) = 1\}$

(b) $\{x \text{ s.t. } d(x,0) < 1\}$

using ℓ^∞ metric.

Soln: (a) 

(b) 

Summary: We've seen four metrics on $\mathbb{R}^n$!

Notation

$$d_{std}(x,x') = \sqrt{\sum_{i=1}^{n} (x_i' - x_i)^2}$$

(standard metric)

$$d_{taxi}(x,x') = \sum_{i=1}^{n} |x_i' - x_i|$$

(taxicab metric)

$$d_{\ell^\infty}(x,x') = \max_{i=1,\dots,n} |x_i' - x_i|$$

(ℓ^∞ metric)

$$d_{discrete}(x,x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x' \end{cases}$$

(discrete metric)

Project for today

Let $X = Y = \mathbb{R}^2$, and let

$$f : X \longrightarrow Y$$

be the identity function. (That is, $f(x) = x$.)

For which metrics d_X and d_Y on X and Y (from the four metrics we've discussed) is f continuous?

⚠ When you do NOT yet have an intuition, definitions are all you've got to go on.

(Today, you probably have very little intuition for these new metrics!)