Lecture 4

4.1 Some confusions

Students have come in with some confusions, so let me discuss them.

1. The notation
$$(X, d_X)$$
does *not* refer to some point on $\mathbb{R}^2$. It is a pair of things, but it is not a pair of numbers. For example, X is a set—perhaps the set of bananas in this room—and d_X is a metric on this set.

2. When I write something like
$$f : X \to Y \tag{4.1.1}$$
I mean a function from X to Y, and I have called the function f. Now, when I have chosen two metric spaces, I will often write
$$f : (X, d_X) \to (Y, d_Y). \tag{4.1.2}$$
The two notations (4.1.1) and (4.1.2) mean the same thing. That is, the latter notation still encapsulates a function from X to Y. The reason I include the metrics d_X and d_Y in the notation is because it is often important to remind the reader which metrics we are considering.[1]

3. Make sure you understand my notation using $x \in \mathbb{R}^n$ and the indices x_i. A point $x \in \mathbb{R}^n$ is determined by a finite collection
$$(x_1, x_2, \ldots, x_n)$$

[1] As we have seen, the same set may allow many different metrics.

of real numbers. Likewise, a point $x' \in \mathbb{R}^n$ is determined by a collection

$$(x'_1, x'_2, \ldots, x'_n)$$

of real numbers. So when I write a formula like

$$d_{l^\infty}(x, x') = \max_{i=1,\ldots,n} |x'_i - x_i|$$

I mean that I consider the collection of real numbers[2]

$$\{|x'_1 - x_1|, |x'_2 - x_2|, \ldots, |x'_n - x_n|\}$$

and I take the maximum number in this set (consisting of n real numbers). As an example, if $n = 4$ and I take two points

$$x = (\pi, \sqrt{2}, 1, 1)$$

and

$$x' = (8, 2, \sqrt{3}, 1)$$

then we would have

$$d_{l^\infty}(x, x') = \max\{|8 - \pi|, |2 - \sqrt{2}|, |\sqrt{3} - 1|, |1 - 1|\}$$

so

$$d_{l^\infty}(x, x') = 8 - \pi.$$

4.2 Last time

Last time we talked about the continuity of the following functions:

1. $\mathrm{id} : (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{std})$.
2. $\mathrm{id} : (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{taxi})$.
3. $\mathrm{id} : (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{discrete})$.

The main toolkit I wanted you to come away with was the following:

To investigate the continuity of a function $f : X \to Y$, you have to see if you can fit some open ball (of radius $\delta > 0$) into the preimage of an open ball (of radius $\epsilon > 0$).[3]

[2] Again, the x'_i are the coordinates of the point x', and the x_i are the coordinates of the point x

[3] The former is centered at $x \in X$, the latter is centered at $f(x) \in Y$.

4.3 Open sets of a metric space

4.3.1 Open balls

Today, I want to get us used to talk about open sets. Let's record as a formal definition some of the terms we've been using.

Definition 4.3.1.1. Let (X, d_X) be a metric space. Fix $x \in X$ and real a number $r > 0$.

The *open ball* of radius r, centered at x, is the set

$$\{x' \in X \text{ such that } d_X(x, x') < r.\}$$

We will use any of the following notations[4] to denote this set:

$$\text{Ball}(x; r) \qquad B(x; r) \qquad B_{d_X}(x; r) \qquad B_X(x; r).$$

Remark 4.3.1.2. Some authors extend the definition to the case $r = 0$. Then the open ball of radius 0 is the empty set; but we will not follow this convention.

4.3.2 Definition of open subset

Definition 4.3.2.1 (Open set of a metric space)**.** Let (X, d_X) be a metric space, and let $A \subset X$ be a subset.

We say that A is *open* if it can be written as the union of open balls.

4.3.3 Examples of open sets

Example 4.3.3.1. Let (X, d_X) be a metric space. Fix $x \in X$ and $r > 0$. Then $A = \text{Ball}(x; r)$ is an open set; it can be written as a union of a single open ball—namely $\text{Ball}(x; r)$.

The following is one of the more confusing examples for people.

Example 4.3.3.2. Let (X, d_X) be a metric space and let $A = \emptyset \subset X$ be the empty set.

Then A is open.

[4] Having all this notation is confusing, but you could already see why it was useful to have notation like d_X and d_Y to distinguish different metrics; these riffs/modifications to the notations also come in handy when disambiguating certain metric spaces.

Remark 4.3.3.3 (Digression into unions)**.** Let me explain this a little bit. When you think of unions of sets, you may think of something pictorial like a Venn diagram.

(Draw a Venn diagram.)

How do you take a union of sets? Well, you first specify the sets you want to take the union of, and then you combine their elements into a single set.

What if you specify no sets at all? Then the union of this collection of sets (the empty collection) is the empty set.

Some people are confused by Example 4.3.4.2 because they are only used to seeing unions of some non-zero number of sets.

But let me reiterate: A union of zero-many sets is the empty set. So the empty set can be written as the union of a collection (albeit an empty collection) of open sets.

Example 4.3.3.4. Let (X, d_X) be a metric space. Then the set $A = X$ itself is open.

To see this, fix any $r > 0$, and consider the collection

$$\{\mathrm{Ball}(x; r)\}_{x \in X}.$$

This is a collection of open balls. Let's explain the notation. The curly brackets $\{...\}$ means we are defining a set. The subscript $x \in X$ means for every $x \in X$, we can specify an element in this set. Which element? The notation $\mathrm{Ball}(x; r)$ means that the open ball $\mathrm{Ball}(x; r)$ is the element.

Confusingly, this is an example of a set of sets. You will get used to this.

Now, consider the union

$$\bigcup_{x \in X} \mathrm{Ball}(x; r).$$

This is a potentially gigantic union. There are many sets we are taking the union of—-for every $x \in X$, we are considering the open ball of radius r, and we are taking the union of every single one of these balls.

Note that this union is contained in X, as each ball is a subset of X. Moreover, any element of X is contained in the union, as any $x \in X$ is contained in the ball $\mathrm{Ball}(x, r)$. Thus,

$$X = \bigcup_{x \in X} \mathrm{Ball}(x; r).$$

So X is open (as it is written as a union of open balls).

Example 4.3.3.5. There is an even larger collection one can write down to prove that X is open. Consider the collection

$$\{\mathrm{Ball}(x,r)\}_{x\in X, r>0}$$

where now we are considering an open ball not just for every choice of $x \in X$, but also for every choice of real number $r > 0$.

Let me discuss a common confusion that this example can illustratively dispel—note that even if $r \neq r'$, the balls $\mathrm{Ball}(x,r)$ and $\mathrm{Ball}(x,r')$ may be the same. (We saw this in the discrete metric; for example, $r = 0.5$ and $r' = 0.4$ give the same open balls.)

Thus, the *subscripts* in the set notation do *not* need to uniquely specify an element of the set.

Another confusion: When exhibiting that a set A is open, you do not need to choose some efficient collection of open balls. For example, we have seen two ways to exhibit X as an open set. The second way we have seen (which not only takes an open ball for every x, but also for every $r > 0$) is far less "efficient" because we have so many open balls; that is fine. Do not be tempted to make a "snug" or "just right" collection of open balls to form a set, as overkill is sometimes useful.

Exercise 4.3.3.6. Let $X = \mathbb{R}^2$ and let A be the set

$$A = \{x \in \mathbb{R}^2 \text{ such that } d_{l^\infty}(0,x) < \delta\}.$$

This is the "open" square centered at the origin of width 2δ. (Put another way, this is the open ball of radius δ in $(\mathbb{R}^2, d_{l^\infty})$. For which of the following metrics on X is A an open set?

1. d_{std}
2. $d_{discrete}$
3. d_{taxi}
4. d_{l^∞}

Proof. All of them!

We must write A as the union of open balls. Thus, for every $x \in A$, we must exhibit some open ball contained in A and containing x.

Let us tackle d_{std} first. Given $x = (x_1, x_2)$, there is a well-defined (standard) distance to the boundary of A. Namely, consider the distances

$$|\delta - x_1|, \qquad |-\delta - x_1|, \qquad |\delta - x_2|, \qquad |-\delta - x_2|.$$

These measure the distance of x from the edges of the square. Note that because A is the open square[5], each of these distances is non-zero. Let r_x be the minimum of the four distances above. Then $\mathrm{Ball}_{d_{std}}(x; r_x)$ is contained iN A. Thus we see that

$$\bigcup_{x \in A} \mathrm{Ball}_{d_{std}}(x; r_x) = A.$$

This verifies that A is an open set in $(\mathbb{R}^2, d_{std})$.

I'll omit the proofs of the others. Note that d_{l^∞} is the "easiest" case because A is already an open ball in that case.

As for the other metrics, you simply need to find an r_x for every x such that the open ball of radius r_x (with respect to the chosen metric) is contained in A. For example, this is easy for the discrete metric—just choose r_x to be anything less than or equal to 1. For the taxicab metric, note that the diamond of corner-to-center length r fits inside the standard ball of radius r, so you could choose the same r_x as for d_{std}. □

Exercise 4.3.3.7. Let $X = \mathbb{R}^2$ and let $A = \{x\}$ consist of a single point. For which of the following metrics on X is A an open set?

1. d_{std}

2. $d_{discrete}$

3. d_{taxi}

4. d_{l^∞}

Proof. Only for the discrete metric. Note that any open ball of positive radius in the other metrics contains at least two points (in fact, any open ball of positive radius contains infinitely many points in any of the non-discrete metrics); but A contains only one point, so A could not contain any open ball of positive radius. In particular, A cannot be written as the union of open balls. □

[5]meaning the "boundary" of A is not part of A

Exercise 4.3.3.8. Let $X = \mathbb{R}$ and let $A = [-3, 3]$ be the closed interval from -3 to 3. For which of the following metrics on X is A an open set?

1. d_{std}
2. $d_{discrete}$
3. d_{taxi}
4. d_{l^∞}

Proof. Only the discrete metric.

To see why A is not open in the other metrics, note that if A can be written as a union of open balls, then in particular, there must be some open ball that contains $3 \in A$, and is contained in$A$.

But in any of the non-discrete metric, if an open ball B of positive radius contains 3, it must also contain some number larger than 3. But such a number is not contained in A. In particular, B could not be contained in A.

Finally, A is open in the discrete metric because we can write

$$A = \bigcup_{x \in A} \mathrm{Ball}_{d_{discrete}}(x; r)$$

for any $r \in (0, 1]$. □

4.4 Open sets and continuity

In math, when you've found a way to translate one sophisticated thing into another, you've discovered something wonderful. We're about to discover something wonderful:

Theorem 4.4.0.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Fix $f : X \to Y$ a function. The following are equivalent:

1. f is continuous.
2. For any open set $V \subset Y$, the preimage $f^{-1}(V)$ is open.

4.4.1 Centering open balls

The proof will be streamlined if we utilize the following lemma:

Lemma 4.4.1.1. Let (X, d_X) be a metric space, and fix $x \in X$ and $r > 0$. Suppose that x' is contained in $\text{Ball}_X(x, r)$. Then there exists $r' > 0$ such that

$$\text{Ball}(x', r') \subset \text{Ball}(x, r).$$

In English, if x' is a point contained in an open ball (centered at a possibly different point x), then one can always find an open ball *centered at* x' contains in the original open ball.

Proof. We set

$$r' = r - d_X(x, x').$$

Indeed, if any other point w is contained in $\text{Ball}(x', r')$, the triangle inequality says

$$d_X(x, w) \leq d_X(x, x') + d_X(x', w)$$

but the righthand side satisfies

$$d_X(x, x') + d_X(x', w) < d_X(x, x') + r' = d_X(x, x') + r - d_X(x, x') = r.$$

So we are finished. □

4.4.2 Proof of Theorem ??.

Recall that to prove two statements are equivalent, we need to prove that one implies the other, and vice versa.

Proof. We first prove that (1) implies (2). Let $V \subset Y$ be open. We must prove that if f is continuous, then $f^{-1}(V)$ is open. So choose $x \in f^{-1}(V)$. The goal is to find some ball B_x of positive radius containing x and contained in $f^{-1}(V)$. (If we can do this for all $x \in f^{-1}(V)$, then we have that

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} B_x$$

and this would show that $f^{-1}(V)$ is open.)

Since $V \subset Y$ is open, it can be written as a union of open balls. In particular, there is some open ball $\text{Ball}_Y(y, r)$ such that

$$f(x) \in \text{Ball}_Y(y, r) \subset V.$$

Claim One: There exists some $\epsilon > 0$ such that $\text{Ball}_Y(f(x), \epsilon) \subset \text{Ball}_Y(y, r)$. Indeed, this is the reason I introduced Lemma **??**. Using that lemma, Claim One is proven.

Now, by the continuity of f, there exists δ such that $d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon$; in particular, there is some δ such that the ball of radius δ centered at x is contained in $f^{-1}(\text{Ball}_Y(f(x), \epsilon))$. But we know that

$$\text{Ball}_Y(f(x), \epsilon) \subset \text{Ball}_Y(y, r) \subset V$$

so in particular,

$$\text{Ball}_X(x, \delta) \subset f^{-1}(V).$$

We have accomplished our goal. This proves that (1) implies (2).

Now let us prove that (2) implies (1). If we assume that the preimage of any open set is open, we must prove that f is continuous.

So fix $x \in X$ and fix $\epsilon > 0$. Then the open ball $V = \text{Ball}_Y(f(x), \epsilon)$ is an open subset of Y, so in particular, we know that $f^{-1}(V)$ is an open subset of X. By definition, then, it can be written as the union of open balls—in particular, there is some open ball containing x. By Lemma **??**, we may choose this ball to be centered at x, and we will write its radius as δ. By construction, this ball is contained in $f^{-1}(V)$, and we thus have

$$f(\text{Ball}_X(x, \delta)) \subset V = \text{Ball}_Y(f(x), \epsilon).$$

In other words, if any point x' is within δ of x, it follows that $d_Y(f(x), f(x')) < \epsilon$. This completes the proof. □