Lecture 7

Tuesday, September 17th

7.1 Non-negativity of metrics

Exercise 7.1.0.1. Let (X, d) be a metric space. Show that $d(x, x') \geq 0$ for any $x, x' \in X$.

Proof. Use the triangle inequality for $x = x' = x''$. Then

$$0 = d(x, x'') \leq d(x, x') + d(x', x'') = 2d(x, x').$$

So (dividing the beginning and the end by 2), we see $d(x, x') \geq 0$. $\square$

7.2 Simplifying the verification of continuity

Exercise 7.2.0.1. Let (X, d_X) and (Y, d_Y) be metric spaces, and fix a function $f : X \to Y$. Show the following are equivalent:

1. For any open set V, $f^{-1}(V)$ is an open set.

2. For any open ball $\text{Ball}_\epsilon(y)$, $f^{-1}(V)$ is an open set.

Proof. 1 implies 2: Any open ball is an open set; so setting $V = \text{Ball}_\epsilon(y)$, 1 implies 2.

2 implies 1: We know that any open set V is a union of open balls, so

$$V = \bigcup \text{Ball}_\epsilon(y)$$

for some collection of open balls. Thus

$$f^{-1}(V) = \bigcup f^{-1}(\mathrm{Ball}_\epsilon(y))$$

where the righthand side is a union of open subsets of X. In homework you proved that any union of open subsets is again open. Thus $f^{-1}(V)$ is open. This proves 2 implies 1. □

Putting together everything, we have proven the following so far in this class:

Theorem 7.2.0.2. Let (X, d_X) and (Y, d_Y) be metric spaces. Fix a function $f : X \to Y$. The following are equivalent:

1. f is continuous.
2. The preimage of any open subset of Y is an open subset of X.
3. The preimage of any open ball of Y is an open subset of X.
4. f sends convergent sequences in X to convergent sequences in X.

7.3 From homework

Let X and Y be metric spaces. Define the following metric on $X \times Y$:

$$d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

Show that the projection map $(x, y) \mapsto x$ is continuous.

Proof. There are two ways you could do this.

(i) Using ϵ-δ. Fix an element of the domain, (x_0, y_0), and some $\epsilon > 0$. We must show the existence of some δ such that

$$d((x_0, y_0), (x, y)) < \delta \implies d_X(x_0, x) < \epsilon. \tag{7.3.1}$$

I claim any $\delta \leq \epsilon$ works. This is because

$$d((x_0, y_0), (x, y)) = d_X(x_0, x) + d_Y(y_0, y) \geq d_X(x_0, x).$$

Where the last inequality follows because $d_Y(y_0, y) \geq 0$. Thus (7.3.1) follows if $\delta \leq \epsilon$.

(ii) Using that a function is continuous if and only if the preimage of any open set is open.

Let's call our function π, so $\pi(x, y) = x$. By the exercise earlier this lecture, we must verify that for any open ball in X, the preimage is an open subset of X.

So fix an open ball $B_\epsilon(x) \subset X$. Here, $x \in X$ and $\epsilon > 0$. By definition of π, the preimage of this is the set of all pairs (x, y) such that x is in the ball, and y is arbitrary. This can be written as a union of open balls as well:

$$\bigcup_{r,x' \text{ such that } B_r(x') \subset B_\epsilon(x)} \bigcup_{y \in Y} B_r((x', y))$$

□

Remark 7.3.0.1. Note that (ii) seems a little bit more complicated. Regardless, we have two very different-looking proofs of the same fact. This is a good sign that the equivalent criteria for continuity are appreciably different, and hence useful! (Having two very different ways to tackle the same problem is a gift.)

7.4 Intuition for open sets in metric spaces

What is the intuition for how to think about an open set in a metric space? Recall that an open set in a metric space is any subset that can be written as a union of open balls. Recall also that we proved the following lemma last time I lectured: If x is contained in some open ball $\text{Ball}_{\epsilon'}(x')$, then there is another open ball $\text{Ball}_\epsilon(x)$, centered at x, taht is contained in $\text{Ball}_{\epsilon'}(x')$.

Corollary 7.4.0.1 (Of the Lemma)**.** Let (X, d) be a metric space and let $U \subset X$ be an open subset. Then for any $x \in U$, there exists an open ball centered at x.

In fact, we have

Proposition 7.4.0.2. Let (X, d_X) be a metric space and fix a subset $U \subset X$. The following are equivalent:

1. U is an open subset.

2. For any $x \in U$, U contains an open ball of some (small) positive radius centered at x. That is, there exists $\delta > 0$ so that $\text{Ball}(x; \delta) \subset U$.

Proof. 1 implies 2. We use the (re)centering lemma from last time I lectured. If U is open, it's a union of open balls:

$$U = \bigcup_{\alpha} \mathrm{Ball}(x_\alpha, \delta_\alpha)$$

where α indexes some collection of centers x_α and radii δ_α. Thus for any $x \in U$, there is some α so that $x \in \mathrm{Ball}(x_\alpha, r_\alpha)$. By the (re)centering lemma, this means that there is some δ so that

$$\mathrm{Ball}(x, \delta) \subset \mathrm{Ball}(x_\alpha, r_\alpha).$$

In particular,

$$\mathrm{Ball}(x, \delta) \subset U.$$

2 implies 1. For every $x \in U$, choose δ_x so that $\mathrm{Ball}(x, \delta_x) \subset U$. Then we have that

$$\bigcup_{x \in U} \mathrm{Ball}(x, \delta_x) = U.$$

To see this equality, note that the righthand side is contained in U (because a union of subsets is still a subset). The lefthand side is contained in the righthand side: Given $x' \in U$, note that $\mathrm{Ball}(x', \delta_{x'})$ is one of the balls in the union on the lefthand side, and in particular, $x' \in \mathrm{Ball}(x', \delta_{x'})$. □

Remark 7.4.0.3. This proposition is supposed to give you intuition for what open sets look like: U is open if and only if for any $x \in U$, x has "enough wiggle room," or "enough breathing room" in U. By "wiggle room," I mean there is some δ so that x can move around in an open ball of radius δ without leaving U.

Warning 7.4.0.4. The notion of being an open subset depends on the metric space we are in. That is, when we say "U is open," we have a metric space in mind of which U is a subset.

Example 7.4.0.5 (Of sets that are not open)**.** Let $A \subset \mathbb{R}$ be a closed bounded interval. Then A is not open. For example, at the endpoint x of A, no open interval about x is fully contained in A. (To see this: Any open interval $(x - \epsilon, x + \epsilon)$ contains some element larger than, or some element less than, x. But because x is an endpoint, it is either the minimal or maximal element of A. Without loss of generality, assume x is minimal. Then A could not contain an element less than x itself.)

Likewise, let $A \subset \mathbb{R}^2$ be a closed bounded interval. Then A is not open. For example, even if x is in the interior of A, no open ball of $\mathbb{R}^2$ fits inside A.

7.5 Beyond metric spaces

So just as we're getting used to metric spaces, I want to suggest to you that the zoo of metric spaces is too constricting. For the next half an hour, I'd like you to think about the following problems:

1. Can you give the circle a metric space structure? How about the sphere? How? Can you give any subset of $\mathbb{R}^n$ a metric space structure? Are they meaningful?

2. Consider the "set of all lines through the origin" in $\mathbb{R}^2$. Make sure you think about what this means. Can you give this a metric space structure?

3. Consider the shape you would get if you were to take a sheet of paper, and carefully glue/tape two opposing edges together. Can you give this a metric space structure?

What we saw in class is that the first example is not so bad to tackle: Any subset $A \subset X$ of metric space (X, d_X) can be given a metric space structure.

Definition 7.5.0.1. Let (X, d_X) be a metric space and let $A \subset X$ be a subset. The *subset metric*, or *induced metric* on A is

$$d_A(x, x') := d_X(x, x').$$

The subset metric is indeed a metric on A. I'll the proof to you as an exercise.

In class we had some difficulty with the set of lines in $\mathbb{R}^2$. We had the insight to try to assign to each line an "angle," but this assignment didn't seem continuous. And depending on how we made the shape obtained by gluing a sheet of paper along its edges, the metrics could be different.

But the notion of having "wiggle room"—we were supposed to discover—is one we can articulate better.

Main idea: Sometimes, it's easier to think about wiggle room (open sets) than it is to think about metrics.

We'll begin next time with topological spaces.