Lecture 8

Thursday, September 19th

8.1 Some announcements

8.1.1 Collaboration policy

Some of your homeworks are far too similar. Please read the collaboration policy I've put online. In short, you can collaborate, and you can consult sources, but when you are *writing* (whether on a laptop, phone, or paper) your homework for submission, you must be alone and not using any resources.

8.1.2 Multiple choice

From now on, multiple choice responses for homework will be submitted online. Links will be on the website every week. Don't let the convenient format fool you—the multiple choices are often the hardest part of the homework. You do not need to hand in paper submissions. These are always do before 1:50 PM on Tuesdays. Anything submitted after 1:50 PM will not be accepted.

8.1.3 Next homework

For the next proof homework, I will scan copies of your submissions and share them with the class. You will get to see the work of other classmates; and your classmates will see your submissions, too.

Put your names on the homeworks; I will anonymize them the best I can when I share with class.

8.2 More on open sets

Last time we saw the following intuition for open sets in metric spaces: A subset $U \subset X$ is open if and only if for every $x \in U$, there is a (small) open ball of positive radius, centered at x and contained in U. We interpreted this to mean that a set U is open if and only if every $x \in U$ has "wiggle room" inside U.

Remark 8.2.0.1. In class discussions and in homeworks, I have also seen some of you engage with the notion of a "boundary" of a set. We will talk about this in due time.

A philosophy I've mentioned more than once: To study objects, we need to study the functions between them. This philosophy is not at all obvious in your earliest serious math classes, but you've at least seen that there are many interesting functions $f : \mathbb{R} \to \mathbb{R}$ to explore (in calculus class, for example). But it is an important philosophy regardless.

What we have seen so far in class—though you may not have noticed it—is that for a map $(X, d_X) \to (Y, d_Y)$ to be continuous certainly depends on the metrics in play, but that it doesn't depend on the entire data of the metrics. For example, we have seen that to check whether a function is continuous, we only need to check whether preimages of open sets are open.

In other words: *Continuity depends only on open sets.*

Remark 8.2.0.2. Combining our intuition of "wiggle room" with our "open set" test for continuity, we arrive at the following intuition. A function $f : X \to Y$ is continuous if and only if: Every $x \in X$ has wiggle room[1] to stay within any specified wiggle room[2] of $f(x)$.

So here's a natural question: Does the collection of open sets of a metric space "remember" the metric of the metric space? Put another way, if (X, d_X) is a metric space, and $\mathcal{T}$ is its collection of open sets, does $\mathcal{T}$ determine d_X? The answer is no:

Theorem 8.2.0.3. Let $\mathcal{T}_{std}$ denote the collection of open sets in $\mathbb{R}^n$ for d_{std}.

Likewise, we let $\mathcal{T}_{taxi}$ and $\mathcal{T}_{l^\infty}$ denote the collections of open sets in $\mathbb{R}^n$ with respect to d_{taxi} and d_{l^∞}.

[1] i.e., a ball of radius δ

[2] a ball of radius ϵ

Then

$$\mathcal{T}_{std} = \mathcal{T}_{taxi} = \mathcal{T}_{l^\infty}.$$

Proof. I will sketch a proof.

Let $U \subset \mathbb{R}^n$ be open with respect to the standard metric. Then for any $x \in U$, there is some δ_{std} so that

$$\mathrm{Ball}_{std}(x; \delta_{std}) \subset U.$$

But given δ_{std}, I claim there exists δ_{taxi} such that

$$\mathrm{Ball}_{taxi}(x; \delta_{taxi}) \subset \mathrm{Ball}_{std}(x; \delta_{std}).$$

That is, any open ball in the standard metric (centered at x) contains an open ball in the taxicab metric (also centered at x). I will just draw a picture of this in class. (It turns out you could take $\delta_{taxi} = \delta_{std}$ because the open ball in the standard metric is convex.)

We conclude that if U is open with respect to d_{std}, it is open with respect to d_{taxi}.

Conversely, let U be open with respect to d_{taxi}. Then for any $x \in U$, there is an open ball $\mathrm{Ball}_{taxi}(x, \delta_{taxi})$ centered at x and contained in U. You can check that the shortest standard distance from x to a "wall" of this taxicab-ball (which is a diamond-shaped region) is given by

$$\delta_{taxi}\sqrt{1/n}$$

where n is the dimension of $\mathbb{R}^n$. Thus we see that

$$\mathrm{Ball}_{std}(x, \delta_{taxi}\sqrt{1/n}) \subset \mathrm{Ball}_{taxi}(x, \delta_{taxi}) \subset U$$

so U is open with respect to d_{std} as well.

This shows $\mathcal{T}_{std} = \mathcal{T}_{taxi}$.

A similar proof shows that $\mathcal{T}_{std} = \mathcal{T}_{l^\infty}$. □

Thus, although the metrics on $\mathbb{R}^n$ are distinct, they give rise to the *same* collection of open sets.

This proves the following:

Corollary 8.2.0.4. The identity functions

$$\begin{array}{rr}
(\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{taxi}) & (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{l^\infty}) \\
(\mathbb{R}^n, d_{taxi}) \to (\mathbb{R}^n, d_{std}) & (\mathbb{R}^n, d_{taxi}) \to (\mathbb{R}^n, d_{l^\infty}) \\
(\mathbb{R}^n, d_{l^\infty}) \to (\mathbb{R}^n, d_{taxi}) & (\mathbb{R}^n, d_{l^\infty}) \to (\mathbb{R}^n, d_{std})
\end{array}$$

are all continuous.

Proof. The preimage of U is given by U. Moreover, if U is open with respect to one of the metrics above, it is also open with respect to any of the others by the previous result. This shows that the preimage of any open subset is open, hence the identity function is continuous. □

Corollary 8.2.0.5. Let (Y, d_Y) be a metric space. Let $f : \mathbb{R}^n \to Y$ be a function. Then f is continuous with respect to the standard (or taxi, or l^∞) metric if and only if it is continuous with respect to any of three metrics above (standard, taxi, or l^∞).

Proof. Let $V \subset Y$ be open. Then $f^{-1}(V)$ is open with respect to one of the three metrics above if and only if it is open with respect to all of them. □

Remark 8.2.0.6. This is the first hint that a notion of distance *helps* detect continuity, but continuity does not depend on a notion of distance! How great is that?

8.3 Constructing new spaces

So we have seen two kinds of things with the word "space" in the name. Let me recall them both:

Definition 8.3.0.1. A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function satisfying:

(0) For all $x, x' \in X$, $d(x, x') = 0 \iff x = x'$.

(1) For all $x, x' \in X$,$d(x, x') = d(x', x)$, and

(2) For all $x, x', x'' \in X$, we have that

$$d(x, x') + d(x', x'') \geq d(x, x'').$$

Definition 8.3.0.2. A *topological space* is a pair

$$(X, \mathcal{T})$$

where X is a set, and $\mathcal{T}$ is a collection of subsets of X, satisfying the following three properties:

1. Both $\emptyset$ and X are elements of $\mathcal{T}$.

2. If $U_1, \ldots, U_k$ is a finite collection of elements of $\mathcal{T}$, then the intersection $U_1 \cap \ldots \cap U_k$ is an element of $\mathcal{T}$. That is, $\mathcal{T}$ is closed under finite intersections.

3. If $\mathcal{A}$ is an arbitrary set and $\mathcal{A} \to \mathcal{T}$ is a function (so for every $\alpha \in \mathcal{A}$ we have an element $U_\alpha \in \mathcal{T}$) then the union
$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$
is also in $\mathcal{T}$. That is, $\mathcal{T}$ is closed under arbitrary unions.

Definition 8.3.0.3. We will call $\mathcal{T}$ a *topology* on X, and any element $U \in \mathcal{T}$ will be called an *open* subset of X.

Example 8.3.0.4. Let (X, d_X) be a metric space, and let $\mathcal{T}$ be the collection of open sets determined by d_X. (That is, $U \in \mathcal{T}$ if and only if U is a union of open balls.) Then you proved in homework that $(X, \mathcal{T})$ is a topological space.

Definition 8.3.0.5. Let (X, d_X) be a metric space and let $\mathcal{T}$ be the collection of open sets with respect to d_X. We say that $\mathcal{T}$ is the topology *induced* by the metric.

Remark 8.3.0.6. Note that these definitions have incredibly different flavors. For example, the notion of metric space depends very much on *numerical* or *quantitative* statements—meaning we rely on properties of the real line. (For example, we rely on the fact that we know how to add elements of $\mathbb{R}$, and on the fact that we knowhow to compare the sizes of elements of $\mathbb{R}$.)

In contrast, the definition of topological space is much more barren—it does not even need mention of the real line. It only relies on the fact that we can consider subsets of a set X, and that we can take unions and intersections of subsets.

Remark 8.3.0.7. This barrenness is both a strength and downside of the definition. The downside is that it takes a lot to get used to. But the strength is that one can speak of many interesting phenomena under the same umbrella—even if we cannot measure distances. In some sense, it frees us from our dependence on distance.

Though I have not framed things this way, we have seen that we can construct new metric spaces from old ones. For example:

1. If (X, d) is a metric space, then any subset $A \subset X$ can be made into a metric space under the subset metric.

2. If (X, d_X) and (Y, d_Y) are metric spaces, then the product $X \times Y$ can be made into a metric space.

Warning 8.3.0.8. There are many non-equivalent ways to make $X \times Y$ into a metric space. This was explored a little bit in one of the exercises in-class; and it already visible in the case of $X = Y = \mathbb{R}$.

For example, $\mathbb{R}^2$ has many different metrics, as we've seen. In parallel, $X \times Y$ can be given any of the following metrics:

1. $d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$. (If $X = Y = (\mathbb{R}, d_{std})$, this gives rise to the taxicab metric in $\mathbb{R}^2$.)

2. $d((x, y), (x', y')) = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}$. (If $X = Y = (\mathbb{R}, d_{std})$, this gives rise to the standard metric in $\mathbb{R}^2$.)

3. $d((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}$. (If $X = Y = (\mathbb{R}, d_{std})$, this gives rise to the l^∞ metric in $\mathbb{R}^2$.)

Remark 8.3.0.9 (Quotients will become easy)**.** Last time, we saw an example where given $S^1 \subset \mathbb{R}^2$, it was very easy to construct a metric on S^1 by just using the subset metric. But when we had to think about the set of lines in $\mathbb{R}^2$, or the cylindrical shape formed by gluing two edges of a sheet of paper together, it was not so obvious how to define a metric that everybody agreed on.

These latter two examples are examples of "quotient spaces." It turns out that while it is very difficult to naturally put metrics on quotient spaces, it is very easy to put a topology on them.

For now, let's see that it is easy to construct topologies on subsets and on product sets, just as it was easy to construct metrics on them. If you believe that quotients are also easy places to construct topologies, we see that working with topological spaces has a lot of pros:

1. It's easy to construct new spaces, and

2. The notion of continuity can be expressed purely in terms of open sets.

8.3.1 Subset topology

Exercise 8.3.1.1. Let $(X, \mathcal{T})$ be a topological space, and fix a subset $A \subset X$. Define

$$\mathcal{T}_A$$

to consist of those subsets $W \subset A$ such that $W = U \cap A$ for some $U \in \mathcal{T}$. (That is, a subset of A is declared open if and only if it is the intersection of A with an open set of X.)

Prove that $\mathcal{T}_A$ is a topology on A.

Proof. We must verify the three properties:

1. $\emptyset \in \mathcal{T}_A$ because $\emptyset \cap A = \emptyset$ and $\emptyset \in \mathcal{T}$. Likewise, $A \in \mathcal{T}_A$ because $A = X \cap A$ and $X \in \mathcal{T}$.

2. Consider a finite collection $W_1, \ldots, W_k \in \mathcal{T}_A$. For each W_i, we know

   $$W_i = U_i \cap A$$

   for some $U_i \in \mathcal{T}$. Then

   $$W_1 \cap \ldots \cap W_k = (U_1 \cap A) \cap \ldots (U_k \cap A) = (U_1 \cap \ldots \cap U_k) \cap A$$

   and this last term is an intersection of an open set $U_1 \cap \ldots \cap U_k$ with A. (Note that the intersection of the U_i is open because $\mathcal{T}$ is a topology.)

3. Now fix an arbitrary collection $\{W_\alpha\}_{\alpha \in \mathcal{A}}$. Then for each α, there exists some $U_\alpha \in \mathcal{T}$ such that $W_\alpha = U_\alpha \cap A$. So

   $$\bigcup_\alpha W_\alpha = \bigcup_\alpha (U_\alpha \cap A) = (\bigcup_\alpha U_\alpha) \cap A.$$

   The set in the parentheses is open because $\mathcal{T}$ is a topology; hence this intersection is in $\mathcal{T}_A$ by definition of $\mathcal{T}_A$.

$\square$

Definition 8.3.1.2. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ a subset. The topology $\mathcal{T}_A$ on A is called the *subset topology* on A.

Remark 8.3.1.3. The above proof is typical of the kinds of proofs you'll see in general topology—the formulas are formulas involving intersections and unions of sets, and the way we index these intersections and unions take a bit of getting used to. This is inherent in the definition of topological space: Because the definition only uses tools of sets (intersections, unions, et cetera) so too will the proofs use such tools.

This is contrast to metric spaces, where we got to use real numbers, additions, and inequalities.

Example 8.3.1.4. Let $X = \mathbb{R}^2$ with the standard topology (induced by the standard metric—or the taxicab metric, or the l^∞ metric). And let $A = S^1 \subset X$ be the unit circle. Let us endow A with the subset topology.

Then a subset $W \subset A$ is open if and only if it is the intersection of an open set of $\mathbb{R}^2$ with the circle. You should try drawing some examples. For instance, any open interval on the circle is an open subset. The circle itself is an open subset, too.

Note that it is impossible for you to draw *every* open subset of $\mathbb{R}^2$; there are just too many. One of the powers of the definition of topological space is that you don't need to know what all open subsets are. Often, you'll only need to know some basic open subsets that all other open subsets are made of; this will lead us to the notion of a basis for a topology, and we'll see that in a week or two.

Remark 8.3.1.5. Suppose that (X, d_X) is a metric space; then we know that a subset $A \subset X$ inherits a metric space structure. Since (A, d_A) is a metric space, one can induce a topology from the metric.

On the other hand, we have just seen that A can inherit a topology from X (without passing through a metric on A).

It turns out that these two topologies are the same. I'll leave this as an exercise to you.

8.3.2 Product topology

Definition 8.3.2.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let us define

$$\mathcal{T}$$

to be the collection of subsets of $X \times Y$ that can be expressed as unions of sets of the form $U \times V$, where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$.

We call this the *product topology* on $X \times Y$.

Exercise 8.3.2.2. Show that $\mathcal{T}$ is a topology on $X \times Y$.

Proof.

1. The empty set can be written as $\emptyset \times \emptyset$, so the empty set is in $\mathcal{T}$. Like wise, $X \times Y$ is in $\mathcal{T}$ because X and Y are open sets of X and Y, respectively.

2. We first note that if U, U' and V, V' are open subsets of X and Y, respectively, then
$$(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V').$$
To see this, note that (x, y) is in the intersection if and only if $x \in U \cap U'$ and $y \in V \cap V'$.

   So suppose that $W \in \mathcal{T}$, so
$$W = \bigcup_{\alpha \in \mathcal{A}} U_\alpha \times V_\alpha$$
is some union of products of open subsets of X and Y. Fix another open subset
$$W' = \bigcup_{\beta \in \mathcal{B}} U_\beta \times V_\beta.$$
Then[3]
$$W \cap W' = \bigcup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} (U_\alpha \times V_\alpha) \cap (U_\beta \times V_\beta) = \bigcup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} (U_\alpha \cap U_\beta) \times (V_\alpha \cap V_\beta)$$
is a union of products of open sets.

3. If $\{W_\alpha\}_{\alpha \in \mathcal{A}}$ is some collection of open sets, each W_α can be expressed as a union
$$W_\alpha = \bigcup_{\gamma \in \mathcal{C}_\alpha} U_\gamma \times V_\gamma.$$
Hence
$$\bigcup_{\alpha \in \mathcal{A}} W_\alpha = \bigcup_{\alpha \in \mathcal{A}, \gamma \in \mathcal{C}_\alpha} U_\gamma \times V_\gamma$$
is a union of products of open sets as well. This shows $\bigcup_\alpha W_\alpha \in \mathcal{T}$.

□

[3] This is a careful application of facts about sets. Note that for something to be in the intersection of W and W', it must be contained in some $U_\alpha \times V_\alpha$ and some $U_\beta \times V_\beta$. Likewise, if an element in the intersection of some $U_\alpha \times V_\alpha$ and some $U_\beta \times V_\beta$, it is in $W \cap W'$. In other words, if I take the intersection of $U_\alpha \times V_\alpha$ with $U_\beta \times V_\beta$ for every α, β, and consider the union of these intersections, I recover $W \cap W'$.