Lecture 9

Tuesday, September 24th

9.1 Intro to quotient spaces

In homework, you showed the following:

Fix a topological space $(X, \mathcal{T}_X)$ and a surjection $p : X \to Y$. Then you can give Y a topology. Moreover, this topology satisfies the following property:

If $(Z, \mathcal{T}_Z)$ is another topological space, then a function $f : Y \to Z$ is continuous if and only if the composition $p \circ f : X \to Z$ is continuous.

Remark 9.1.0.1. The power of this statement is that you can check the continuity of f by *checking something between* X *and* Z, not between Y and Z. If you have more information about X then about $(Y, \mathcal{T}_Y)$, this is a very useful technique.

But where do surjections $X \to Y$ come from? There is a natural source: When Y is a quotient of X. Today's goal is to explain this.

In the last couple classes, we've tried to consider the two following sets:

1. The set of all lines through the origin (in $\mathbb{R}^2$), and
2. The cylinder-like gadget one gets by gluing two edges of a sheet of paper together.

I claim that we can realize both sets as the Y in the quotient space construction, and thereby endow these sets with topologies.

Example 9.1.0.2. In the example of the cylinder-like gadget, the sheet of paper surjects onto the cylinder like object. So the sheet of paper is the X and the cylinder-like gadget is the Y.

It is not at obvious what X one can take to surject onto the set of lines in $\mathbb{R}^2$ through the origin.

9.2 Equivalence relations and quotient sets

I want to tell you how to take a set X and "glue" some of its elements together.

Remark 9.2.0.1. This is imprecise, but is meant to give intuition. In what follows, the following expressions will roughly mean the same thing:

1. To "glue" two points of X together.
2. To make two points of X equal.
3. Identifying two points of X.

The mathematical toolkit we have for identifying points of X is called an equivalence relation.

Definition 9.2.0.2. Let X be a set. An *equivalence relation* on X is a choice of subset

$$E \subset X \times X$$

satisfying the following:

(0) (Reflexivity.) For every $x \in X$, the element (x, x) must be in E.

(1) (Symmetry.) For every $x, x' \in X$, if $(x, x') \in E$, then (x', x) is in E.

(2) (Transitivity.) For every $x, x', x'' \in X$, if $(x, x') \in E$ and $(x', x'') \in E$, then $(x, x'') \in E$.

Notation 9.2.0.3. Let $E \subset X \times X$ be an equivalence relation on X. Then we will write

$$x \sim x'$$

and say "x is related to x'" whenever $(x, x') \in E$.

Example 9.2.0.4. In the $\sim$ notation, the above three properties of an equivalence relation may be written as

(0) (Reflexivity.) For every $x \in X$, $x \sim x$.

(1) (Symmetry.) For every $x, x' \in X$, $x \sim x' \implies x' \sim x$.

(2) (Transitivity.) For every $x, x', x'' \in X$, $x \sim x'$ and $x' \sim x''$ implies $x \sim x''$.

Example 9.2.0.5. The prototypical example of an equivalence relation is the *equality* relation. That is,

$$x \sim x' \iff x = x'.$$

In this example, E is equal to the set of all pairs (x, x). (That is, those (x, x') such that $x = x'$.) In terms of the intuition that an equivalence relation tells you which elements to identify, this relation tells you to introduce no new identifications—i.e., you only glue a point to itself, so you are not gluing any non-distinct points together.

Example 9.2.0.6. Another example of an equivalence relation is to glue everything together—i.e., to glue any two points to each other. That is,

$$x \sim x' \text{ for any } x, x' \in X.$$

That is, E is equal to $X \times X$ itself.

Now let us give a name for the set of all points that are identified to each other.

Definition 9.2.0.7. Fix a set X and an equivalence relation $E \subset X \times X$. An *equivalence class* of E is a subset $A \subset X$ satisfying the following:

(0) A is non-empty.

(1) If $x \in A$ and $x \sim x'$, then $x' \in A$.

(2) If $x, x' \in A$, then $x \sim x'$.

Exercise 9.2.0.8. Fix an equivalence relation E on X and let $A_1, A_2 \subset X$ be two equivalence classes. Show that if there exists an element $x \in A_1 \cap A_2$, then $A_1 = A_2$.

Proof. Let $[x]$ be the collection of those $x' \in X$ such that $x \sim x'$. I first claim that if any equivalence class A contains x, then $A = [x]$.

$A \subset [x]$ follows from property (2) of an equivalence class (Definition 9.2.0.7).

$[x] \subset A$ follows from property (1) of an equivalence class.

Thus $A_1 = [x] = A_2$ and we are finished. □

What the above exercise tells us is that any equivalence relation on X *partitions* X. That is, it allows us to write X as a union of subsets called equivalence classes:

$$X = \bigcup A$$

Moreover, if $A \neq A'$, then $A \cap A' = \emptyset$. Thus X is a union of subsets that are *disjoint* from one another.

Definition 9.2.0.9. Let X be a set and $E \subset X \times X$ an equivalence relation on X. Then we let

$$X/\sim$$

and

$$X/E$$

denote the set of equivalence classes of E. That is,

$$X/E = \{A \subset X \text{ such that } A \text{ is an equivalence class.}\}$$

We call X/E the *quotient set* of X (with respect to E).

Remark 9.2.0.10. So for example, the following notations make sense:

$$A \in X/E, \qquad A \subset X, \qquad x \in A.$$

However, the following *does not* make sense:

$$A \subset X/E, \qquad x \in X/E.$$

Note that we have a function

$$q : X \to X/E, \qquad x \mapsto \text{The equivalence class } A \text{ containing } x.$$

We know that every $x \in X$ belongs to some equivalence class because of property (0) of an equivalence relation, and we know that every x belongs to a unique equivalence class because of the exercise—hence the function q is indeed well-defined.

Definition 9.2.0.11. The function $q : X \to X/E$ is called the *quotient map* (with respect to E).

Remark 9.2.0.12. Intuitively, the data of E tells you which points to glue together. X/E is the set one gets after gluing together those points. The quotient map q tells you that a point $x \in X$ goes to the point in X/E resulting from gluing together all those points related to x.

Remark 9.2.0.13. The function q is a surjection. This is by property (0) of equivalence class: Any equivalence class has at least one element in it, hence any A equals $q(x)$ for some x.

9.2.1 The collection of lines through the origin in $\mathbb{R}^2$

We tackled this set a week ago. Let's give this set a name.

Notation 9.2.1.1 ($\mathbb{R}P^1$)**.** We let $\mathbb{R}P^1$ denote the set of all lines through the origin in $\mathbb{R}^2$.

Remark 9.2.1.2. This notation is common in the literature. $\mathbb{R}P^1$ is also called the *real projective line*.

Our goal is to understand whether we can think of $\mathbb{R}P^1$ as a topological space.

How do you *specify* a line through the origin in $\mathbb{R}^2$?

Approach One. Specify a point on the circle. Then there's a unique line that goes through that point and the origin. So there is a function

$$p : S^1 \to \mathbb{R}P^1.$$

Note that this function is not one-to-one; for example, two antipodal points on a circle (i.e., two points given by angle θ and by $\theta\pi$) determine the same line. Regardless, p is a surjection, so we can try to endow $\mathbb{R}P^1$ with the topology you induced on homework.

Approach two. Specifying an equation for the line. Recall that any line can be expressed as the set of pairs x_1, x_2 satisfying the equation

$$ax_1 + bx_2 = c.$$

If the line is to pass through the origin, then we know c must equal zero. Moreover, for the above equation to specify a line, then at least one of a or b

must be non-zero. So any pair $(a, b) \neq (0, 0)$ determines a line $L_{a,b}$ through the origin:

$$(a, b) \mapsto L_{a,b} = \{(x_1, x_2) \text{ such that } ax_1 + bx_2 = 0.$$

This defines another function

$$p : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}P^1, \qquad (a, b) \mapsto L_{a,b}.$$

This is a surjection because every line through the origin is determined by some equation of the form $ax_1 + bx_2 = 0$. However, this function is not an injection.

Exercise 9.2.1.3. Fix a pair (a, b) and (a', b'). Then $L_{a,b} = L_{a',b'}$ if and only if there exists a non-zero real number $t \neq 0$ such that

$$(ta, tb) = (a', b').$$

So we have laid out two approaches. In either approach, we have found a set X together with a surjection $p : X \to \mathbb{R}P^1$. Moreover, intuitively, both these surjections should feel continuous. (Informally: If you wiggle a point in S^1, you are wiggling the line passing through that point. If you wiggle the parameters a and b, you are wiggling the line given by that parameter.) So a natural way to give a topology to $\mathbb{R}P^1$ is by giving it the *quotient topology* induced by the surjections p (as defined in your homework).

9.3 When are two spaces equivalent?

So we have two distinct ways of exhibiting a surjection to $\mathbb{R}P^1$:

1. As a quotient of the sspace $\{(a, b) \text{ such that } (a, b) \neq (0, 0) = \mathbb{R}^2 \setminus \{(0, 0)\}$, and
2. As a quotient of S^1.

So, a priori, we have two *different* topologies on $\mathbb{R}P^1$. Are they the same? Put another way, are the quotient topologies on

$$(\mathbb{R}^2 \setminus \{(0, 0)\}/\sim) \qquad \text{and} \qquad S^1/\sim$$

"equivalent" in some sense?

This brings us to a natural question:

When should we consider two topological spaces to be equivalent?

I want to emphasize a difference between two things being "equal" (or the same) and two things being "equivalent." For example, a set of three bananas is not the same set as a set of three apples. But they can be treated as equivalent for many set-theoretic purposes. The reason is that they have the same "size," or cardinality; that is, the two sets are in bijection.

Put another way, we consider two sets to be equivalent if there exists a bijection between them. And the bijection exhibits in what way we consider them to be equivalent.

So how about spaces? Spaces are not just sets, but sets equipped with a topology (i.e., a collection open sets). So we should consider two spaces to be equivalent if they are not only equivalent as sets, but also have "equivalent" collections of open sets. More on this next time.