Lecture 10

Thursday, September 26th

What we learned last time has a huge pay-off: We get to construct a lot of fun and interesting spaces.

10.1 Elaborations on last time

Last time we talked about equivalence relations and equivalence classes; these allowed us to construct the quotient topology on quotient spaces. Let me introduce a bit of notation:

Notation 10.1.0.1. Let E be an equivalence relation on X, and let $A \subset X$ be an equivalence class containing some $x \in X$. We will write

$$[x] \subset X$$

for this equivalence class. So $[x] = [x']$ if and only if $x \sim x'$.

Example 10.1.0.2. Let $X = \{1, 2, 3, 4\}$ be a set of 4 elements we call 1, 2, 3 and 4. Let's say we want to glue 2 to 3, and 3 to 4. Such a gluing will result in a set with two elements.

Let's be explicit about the equivalence relation $E \subset X \times X$ that encodes the idea that we want to glue 2 to 3 and 3 to 4. E is, explicitly:

$$\begin{aligned} E = \{&(1,1),(2,2),(3,3),(4,4),\\ &(2,3),(3,2)\\ &(3,4),(4,3)\\ &(2,4),(4,2)\}. \end{aligned}$$

The first row of elements are in E by reflexivity: Any element should be related to itself. For example, $(1,1) \in E$ means that $1 \sim 1$.

The second row follows because we want to glue 2 to 3 (so we want $2 \sim 3$) and by symmetry (which says that 3 must be related to 2 — i.e., $(3,2) \in E$).

Likewise for the third row, because we want to glue (i.e., declare equivalent) 3 to 4.

Finally, the last row follows by transitivity: If we are gluing 2 to 3, and 3 to 4, then we are also gluing 2 to 4.

We can list the equivalence classes of E explicitly. We have two:

$$A = \{1\}, \qquad B = \{2,3,4\}.$$

The set $X/E = \{A, B\}$ is the two element-set we obtain by gluing as prescribed.

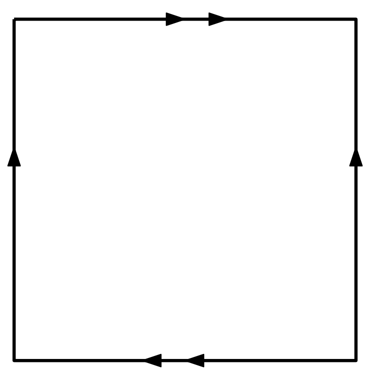
Remark 10.1.0.3. To reiterate: An equivalence relation E tells you what you elements you want to glue together, and the quotient set X/E is the result of gluing.

Let me motivate quotient spaces a little more. You are probably used to visualizing shapes in three-dimensional space (i.e., $\mathbb{R}^3$); but there are spaces that cannot be visualized as sitting in $\mathbb{R}^3$. Here is an example:

Example 10.1.0.4. Let $X = [0,1] \times [0,1] \subset \mathbb{R}^2$; X is a square. Consider the relation

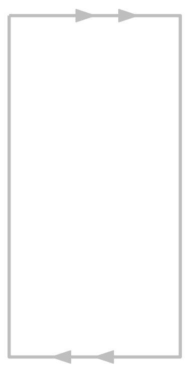
$$(0, x_2) \sim (1, x_2) \qquad \text{and} \qquad (x_1, 0) \sim (1 - x_1, 1)$$

(and of course, (x_1, x_2) is related to itself) for all $x_1, x_2 \in [0,1]$. The common shorthand drawing for this gluing is as follows:



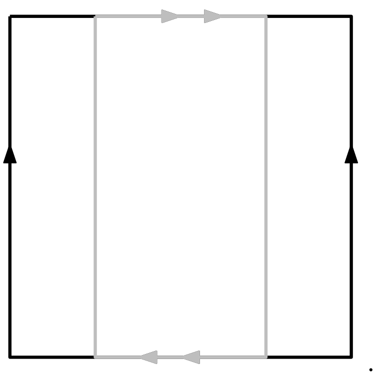
(Note that the vertical edges are glued in a way respecting their orientations, while the two horizontal edges are glued in a way that flips them. You should try and make this shape at home. I promise you won't be able to do it in a way where things are embedded nicely (in fact, embedded at all!) in your three-dimensional space. The upshot is that quotient spaces give us a way of talking about many different kinds of spaces—even those that we cannot visualize in three-dimensions. Note also that (because of our particular presentation) it seems at first difficult to try to embed this space in $\mathbb{R}^n$ for any n.

Remark 10.1.0.5. However, if we only glue the two horizontal edges (with orientations flipped), one can perform this gluing in our three-dimensional space.



Give it a try if you've never seen this before. The result space is called a *Mobius strip*.

Let me also remark that the mobius strip actually embeds into the shape from the previous example:



So even though we cannot visualize easily the shape from the previous example, we do know that it contains a mobius strip (somehow).

So at the very least, we can motivate quotient spaces as follows: Quotient spaces are often examples of interesting spaces.

10.2 Coproducts (disjoint unions)

Here is another way:

Notation 10.2.0.1. Let X and Y be sets. We let

$$X \coprod Y$$

denote the *disjoint union* of X and Y. This is also called the *coproduct* of X and Y.

Remark 10.2.0.2. In case you haven't see $\coprod$ before, let me tell you how it's different from the usual union operation. For example, with the ordinary union, $X \cup X = X$; that is, any set union itself gives back that set.

But $X \coprod X \neq X$. Disjoint union means we formally treat the two sets as made of distinct elements (even if the sets may have intersection) and then take the union. So for example, if X is a finite set with N elements, then $X \coprod X$ is a finite set with $2N$ elements.

A more formal way to construct the disjoint union of sets is as follows. Fix a set $\mathcal{A}$ and for each $a \in \mathcal{A}$, fix a set X_a. Then the disjoint union of the X_a, denote

$$\coprod_{a \in \mathcal{A}} X_a,$$

is the set consisting of those ordered pairs (a, x) where $a \in \mathcal{A}$ and $x \in X_a$. In particular, when $\mathcal{A}$ is a set consisting of two elements a and b, this defines the disjoint union of two sets:

$$X_a \coprod X_b.$$

Definition 10.2.0.3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. We endow the disjoint union

$$X \coprod Y$$

with the following topology: A subset $W \subset X \coprod Y$ is open if and only if $W \cap X$ and $W \cap Y$ are both open.

We call this the *coproduct* topology on $X \coprod Y$.

Example 10.2.0.4. Let $*$ be a set consisting of one element. This has a unique topology—every subset is declared open.

Then $* \coprod *$ is a set consisting of two elements. Every subset of $* \coprod *$ is open.

More generally, for any set $\mathcal{A}$, one can take the coproduct

$$\coprod_{\alpha \in \mathcal{A}} *.$$

This coproduct is a set in bijection with $\mathcal{A}$, and it has a topology such that every subset is open.

Example 10.2.0.5. Let X, Y, Z be topological spaces. Then a function $f : X \coprod Y \to Z$ is continuous if and only if the associated functions $X \to Z$ and $Y \to Z$ are both continuous.

Put another way, to check whether f is continuous, we do not need to check anything that involves both X and Y at once; informally, this means that X and Y do not have topologies that "talk" to each other; a point $x \in X$ has no "desire" or "knowledge" to be within wiggling room of a point of Y.

10.2.1 Discrete spaces

Definition 10.2.1.1. Let X be a set. The *discrete topology* on X is the topology for which every subset of X is open.

A topological space equipped with the discrete topology is called a discrete space.

Example 10.2.1.2. Suppose X and Y are discrete spaces. Then any bijection between them is a homeomorphism.

Example 10.2.1.3. Any discrete space X is homeomorphic to a coproduct; namely, setting $\mathcal{A} = X$, we have that X is homeomorphic to

$$\coprod_{a \in \mathcal{A}} *.$$

Example 10.2.1.4. Let $X = \mathbb{R}^n$ and equip X with the discrete metric. Then the associated topology on $\mathbb{R}^n$ is the discrete topology. To see this, note that for any $x \in \mathbb{R}^n$, the open ball of radius $r < 1$ centered at x is just the set $\{x\}$ consisting only of x. Hence any subset of $\mathbb{R}^n$ is open (because any set is a union of its elements).

Remark 10.2.1.5. You should think of a discrete space as made up of a bunch of "disconnected" points.

For example, continuous maps out of discrete spaces are not interesting from the viewpoint of topology: For any topological space Y, and any discrete space X, any function $f : X \to Y$ is automatically continuous. (This is because the preimage of any $V \subset Y$ is some subset of X, but every subset of X is open!)

Intuitively, a function is not continuous precisely when it doesn't respect some desire for points on X to "stay wiggling near each other." That any function $f : X \to Y$ is continuous means that the points of X have no such wiggling relationship with each other.

10.2.2 The trivial topology

Let X be any set. We saw in the previous section that X admits a topology called the discrete topology; it was in some sense a silly topology because any subset of X was deemed open. There is another silly topology: Declare $\emptyset$ and X to be the only open subsets of X.

Definition 10.2.2.1. Let X be a set. The *trivial topology* on X is the one for which $\emptyset$ and X are the only open sets.

Example 10.2.2.2. Let W be any topological space, and let X be a space equipped with the trivial topology. Then any function $W \to X$ is continuous.

This is "dual" to the discrete topology; it was easy to construct continuous functions whose *domains* were discrete; what we see is that it is also easy to construct continuous functions whose codomains have trivial topology.

Example 10.2.2.3. There are only two sets for which the trivial and discrete topology coincide: The empty set, and the set with one element.

Example 10.2.2.4. Let X be a set, and let $\mathcal{T}_{triv}$ and $\mathcal{T}_{discrete}$ be the trivial and discrete topologies on X, respectively. Then the identity function

$$(X, \mathcal{T}_{discrete}) \to (X, \mathcal{T}_{triv})$$

is continuous, but the identity function

$$(X, \mathcal{T}_{triv}) \to (X, \mathcal{T}_{discrete})$$

is not.

10.3 Summary of how to make topological spaces

So far we've seen:

1. Metric spaces give rise to topological spaces. Explicitly, given (X, d), we declare $U \in \mathcal{T}_X$ if and only if U is a union of open balls. (Example: $(\mathbb{R}, d_{std})$ gives rise to $\mathbb{R}$ with the standard topology—a subset of $\mathbb{R}$ is open if and only if it is a union of open intervals.)

2. Products of topological spaces have natural topological space structures. (Example: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and more generally, $\mathbb{R}^n$.) Given $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, we declared a subset of $X \times Y$ to be open if and only if it is a union of sets of the form $U \times V$ with $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$.

3. Subsets of topological spaces have natural topological space structures. (Example: $S^1 \subset \mathbb{R}^2$).

4. Quotients of topological spaces have natural topological space structures. (Example: $\mathbb{R}P^1$, or the "cylinder"-like object.)

5. Coproducts

We have so many ways of making new spaces, that maybe you'll do some different things and end up making equivalent spaces! But what does it mean for spaces to be equivalent?

10.4 Homeomorphism

Last class we left off on the topic of: When are two spaces equivalent?

Proposition 10.4.0.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Fix a bijection $f : X \to Y$. Then the following are equivalent:

1. The preimage operation f^{-1} sends open sets of Y to open sets of X; moreover the induced function $\mathcal{T}_Y \to \mathcal{T}_X$ is a bijection.

2. f is continuous, and the inverse of f is continuous.

I did not give the proof in class, but you may assume this result from now on. Here is a proof for those who are curious:

Proof. (1) $\implies$ (2): By definition, if the operation $V \mapsto f^{-1}(V)$ sends open sets to open sets, then f is continuous. Let g be the inverse function to f. We must show that g is continuous to finish the proof.

If $U \subset X$ is open, we must verify that $g^{-1}(U)$ is open in Y. Well, the operation $f^{-1} : \mathcal{T}_Y \to \mathcal{T}_X$ is a bijection, and in particular, a surjection. Hence there is some open subset $V \subset Y$ such that $U = f^{-1}(V)$. Because g is the inverse to f, we have that $g^{-1}(U) = f(U) = f(f^{-1}(V))$; because f is a surjection, $f(f^{-1}(V)) = V$. Thus g is continuous, as was to be shown.

(2) $\implies$ (1): Because f is continuous, we know that the preimage operation f^{-1} defines a function $\mathcal{T}_Y \to \mathcal{T}_X$. This is an injection because f is a surjection.[1] To show that $\mathcal{T}_Y \to \mathcal{T}_X$ is a surjection, we invoke that the inverse function g is continuous—for then the preimage operation defines a function $g^{-1} : \mathcal{T}_X \to \mathcal{T}_Y$, and because $f^{-1}(g^{-1}(U)) = U$, we conclude that the preimage operation f^{-1} is a surjection. □

So this gives us the following notion of equivalence of topological spaces:

Definition 10.4.0.2. Let X and Y be topological spaces. We say that a function $f : X \to Y$ is a *homeomorphism* if

1. f is a bijection,
2. f is continuous, and
3. The inverse of f is continuous.

We will say that two topological spaces are *homeomorphic* if there exists a homeomorphism between them, and we will write

$$X \cong Y$$

to mean that X is homeomorphic to Y.

[1] In general, for any function f, we have that $f(f^{-1}(V)) \subset V$. When f is a surjection, $f(f^{-1}(V)) = V$. In particular, if f is a surjection, we have that $f^{-1}(V) = f^{-1}(V') \implies V = V'$.

Example 10.4.0.3. Note that even if f is a bijection and continuous, it may be that the inverse to f is not continuous. An example is given by the identity function

$$f = \mathrm{id} : (\mathbb{R}^n, d_{discrete}) \to (\mathbb{R}^n, d_{std}).$$

This is a continuous function, but its inverse function (which is again the identity function) is not continuous.

10.5 Examples of quotient spaces

Quotient spaces take some time to get used to; but I want to encourage you to think freely and pictorially about them as you also learn to think rigorously about them. Here are examples to give you some intuition.

Example 10.5.0.1. Let $X = [0, 1]$ be the closed interval from 0 to 1. Let E be the equivalence relation where the only non-trivial relation is $0 \sim 1$. We will write

$$X/(0 \sim 1)$$

for the quotient space (equipped with the quotient topology from your homework).

Then

$$X/(0 \sim 1) \cong S^1.$$

That is, this quotient space is homeomorphic to a circle. This is "clear" if you know how to visualize things, but otherwise it can seem like a non-trivial statement. We'll talk about how to prove this at a later time, but you should draw a picture to see why this might be true.