Lecture 11

More on equivalence relations, and $\mathbb{R}P^n$

11.1 Surjections and equivalence relations

A question was asked in class:

"Does any continuous surjection $p : X \to Y$ form an equivalence relation?"

We recognized this was imprecise because we have no definition for when a function "forms" an equivalence relation. For example:

1. Equivalence relation on what? On X or on Y?
2. What relationship would one like p to have with this relation?
3. What does continuity have to do with it?

Indeed, let's take continuity out of the picture—so that X and Y are just sets, and p is just a function. After several rounds of discussion, we came upon the following question:

Question 11.1.0.1. Let $p : X \to Y$ be a surjection. Does there exist an equivalence relation on X so that $X/\sim$ is in bijection with Y?

Here is what we saw:

Proposition 11.1.0.2. Let $p : X \to Y$ be a surjection. And define a relation on X by

$$x \sim x' \iff p(x) = p(x'). \tag{11.1.1}$$

Equivalently, this relation is given by the set $E \subset X \times X$ where

$$E = \{(x, x') \text{ such that } p(x) = p(x')\}.$$

Then this is an equivalence relation.

Proof. (Reflexivity.) We must show that for all $x \in X$, we have $x \sim x$. This follows because $p(x) = p(x)$.

(Symmetry.) We must show that for all $x, x' \in X$, if $x \sim x'$, then $x' \sim x$. Well,

$$x \sim x' \implies p(x) = p(x') \implies p(x') = p(x) \implies x' \sim x.$$

(Transitivity.) We must show that for all $x, x', x'' \in X$, if $x \sim x'$ and $x' \sim x''$, then $x \sim x''$. Here is a proof:

$$x \sim x', x' \sim x'' \implies p(x) = p(x'), p(x') = p(x'') \implies p(x) = p(x'') \implies x \sim s''.$$

□

Then we proved:

Proposition 11.1.0.3. Let $X/\sim$ be the quotient set defined by the equivalence relation (11.1.1). Then there exists a bijection from $X/\sim$ to Y.

Remark 11.1.0.4. Just to make sure we know what's going on:

- The definition of $\sim$ depended on p; we expect the function $X/\sim \to Y$ to also depend in some way on p.
- $X/\sim$ is the set of equivalence classes of $\sim$. that is, it is a set of sets.
- Recall that an equivalence class of $\sim$ is a subset $A \subset X$ such that
  - if $x \in A$, then all x' such that $x' \sim x$ is also in A; moreover,
  - if $x, x' \in A$, then $x \sim x'$
- Recall also that if $x \in A$ and A is an equivalence class, we write

$$[x] = A.$$

Proof. Let's first define the bijection. We will call it $\phi : X/\sim \to Y$.

Given $A \in X/\sim$, let $x \in A$. Then we define

$$\phi(A) := p(x).$$

(Well-definedness of ϕ.) Note that this function ϕ seems to depend on something—that is, to define $\phi(A)$, we first had to *choose* $x \in A$, and then apply p to x. But our function should depend only on A (the element of the domain $X/\sim$) and not on a choice of x. Let us verify this. If we had chosen another $x' \in A$, then—by definition of equivalences class—we know that $x \sim x'$. Hence—by the definition of $\sim$ in (11.1.1)—we know $p(x) = p(x')$. So ϕ is *well-defined.*[1] Which is to say, ϕ is indeed a function with the specified domain and codomain.

(Injection.) We now prove ϕ is an injection. This means we must show that if $\phi(A) = \phi(A')$, then $A = A'$.

So suppose $\phi(A) = \phi(A')$. By definition of ϕ, that means that for all $x \in A$ and $x' \in A'$, we have

$$p(x) = \phi(A) = \phi(A') = p(x').$$

But by definition of our equivalence relation (11.1.1), we know that $p(x) = p(x') \implies x \sim x'$. So $A = A'$ because two equivalences classes that share an element are identical. (This is from a previous class.)

(Surjection.) We now prove ϕ is a surjection. Fix $y \in Y$. Because p is a surjection, there exists $x \in X$ so that $p(x) = y$. Let $A = [x]$ be the equivalence class containing x. Then by definition of ϕ, we have that

$$\phi(A) = \phi([x]) = p(x) = y.$$

This proves that ϕ is a surjection. □

Remark 11.1.0.5. The only place we used that p is a surjection is in proving that ϕ is a surjection. In general, regardless of whether p is a surjection, we will always have that $X/\sim$ is in bijection with the image of p.

Next, we can actually try to ask some question about topology. Namely,

[1]In general, we say that an assignment a priori depending on particular choices is *well-defined* if it does not depend on those choices.

Question. Let X and Y be topological spaces, and fix a continuous surjection $p : X \to Y$. Consider the bijection

$$\phi : X/\sim \to Y$$

from above.

1. Is ϕ continuous?

2. Is ϕ a homeomorphism?

Let's take this one step at a time. Recall:

Definition 11.1.0.6 (Quotient topology.)**.** Let X be a topological space and $\sim$ an equivalence relation on X. Let

$$q : X \to X/\sim \qquad x \mapsto [x]$$

be the quotient map. Then we define a topology on $X/\sim$ by declaring that $U \subset X/\sim$ is open if and only if $q^{-1}(U)$ is open.

Proposition 11.1.0.7. Give $X/\sim$ the quotient topology. Then the quotient map $q : X \to X/\sim$ is continuous.

Proof. We must prove that for any $U \subset X/\sim$ open, $q^{-1}(U)$ is open. This is how openness for a subset of $X/\sim$ is defined. □

Importantly, note that the quotient topology on $X/\sim$ is *exactly* the topology you put (in homework) on the codomain of any surjection. In particular, we have the following result from homework:

Proposition 11.1.0.8. Let $f : X/\sim \to Z$ be a function. Then f is continuous if and only if the composition $f \circ q$ is open.

By the way, going back to the question above: While the map $X/\sim \to Y$ is continuous, it is not always a homeomorphism. For example, let X be $\mathbb{R}^n$ with the discrete topology, and Y be $\mathbb{R}^n$ with the standard topology (induced by the standard metric). Then the identity function $X \to Y$ is continuous, and the map $X/\sim \to Y$ is also continuous (and a bijection). But its inverse is not continuous, because the composite

$$\text{id} : Y \to X/\sim \to X$$

is not a continuous map. (Note that because $\text{id} : X \to Y$ is a bijection, the quotient map $X \to X/\sim$ has is a bijection, hence there is an inverse $X/\sim \to X$.)

11.2 $\mathbb{R}P^1$ and $\mathbb{R}P^2$

Next time, we will talk more about the following two spaces:

$\mathbb{R}P^1$, which is the space of all lines through the origin in $\mathbb{R}^2$. This is topologized by noticing that there is a surjection

$$p : S^1 \to \mathbb{R}P^1$$

which sends a point x on the circle to the unique line passing through x and the origin. p is a surjection (because any line through the origin intersects the circle at some x), and is not an injection, but is a 2-to-1 map (every line through the origin goes through exactly two points on the circle, so for every $L \in \mathbb{R}P^1$, there are exactly two points in $p^{-1}(L)$). Then we can endow $\mathbb{R}P^1$ with the quotient topology.

Likewise, let $\mathbb{R}P^2$ be the space of all lines through the origin in $\mathbb{R}^3$. How is this a space? That is, how do we topologize it?

We do the same trick as before: We notice there is a function $S^2 \to \mathbb{R}P^2$. Given a point x on the sphere, there is a unique line through the origin that also passes through x. We call this assignment $p : S^2 \to \mathbb{R}P^2$. Then p, as before, is a 2-to-1 surjection. We topologize $\mathbb{R}P^2$ by the quotient topology.

This $\mathbb{R}P^2$ is a cool space. It turns out it cannot be embedded into $\mathbb{R}^3$, so we do not have a perfect way of visualizing it. Moreover, we will eventually see that $\mathbb{R}P^2$ admits an embedding of the Mobius band inside of it.

More on this next time.