Lecture 12

Real projective plane

12.1 $\mathbb{R}P^1$ and $\mathbb{R}P^2$

Recall that $\mathbb{R}P^1$ is the set of lines through the origin in $\mathbb{R}^2$. And $\mathbb{R}P^2$ is the set of lines through the origin in $\mathbb{R}^3$. These spaces are pronounced "R P one" and "R P two," respectively.

Remark 12.1.0.1. Somebody asked what this notation stands for.

$\mathbb{R}$ stands for the real numbers.

P stands for the word "projective." This word originates in "projective geometry," which is the study of how the geometry of our world behaves when it's projected onto (for example) a canvas, or our retina.

Sometimes, $\mathbb{R}P^1$ is called the *real projective line*, and $\mathbb{R}P^2$ is called the *real projective space*.

Remark 12.1.0.2. Somebody asked if there is a "complex" version, say $\mathbb{C}P^1$ and $\mathbb{C}P^2$. There are such things. Recall that we have seen that $\mathbb{R}P^1$ can be written as the following quotient set:

$$\{(x_1, x_2) \neq (0,0)\}/\sim, \qquad (x_1, x_2) \sim (x_1', x_2') \iff x_1 = tx_1' \text{ and } x_2 = tx_2' \text{ for some } t \neq 0.$$

Well, we can now pretend that x_1, x_2 are *complex* number, and define a quotient set using the exact same notation as above (with t now also a complex number). This is a quotient of the space $\mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$. Since $\mathbb{C} \cong \mathbb{R}^2$, this is a quotient of the space $\mathbb{R}^4 \setminus \{(0,0,0,0)\}$. We call this quotient $\mathbb{C}P^1$.

It turns out that $\mathbb{C}P^1$ is homeomorphic to the sphere, S^2.

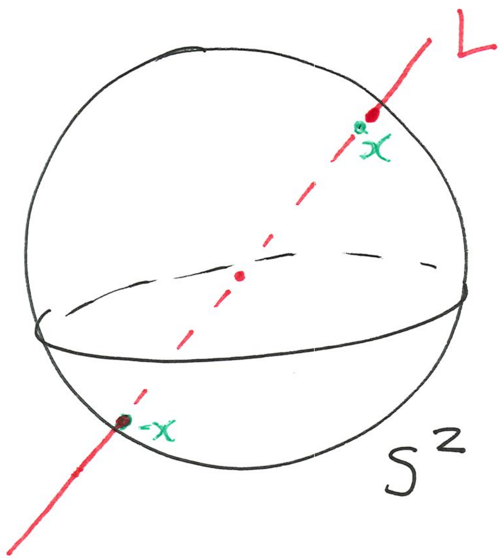
12.2 The topology of $\mathbb{R}P^2$

Today we're going to study $\mathbb{R}P^2$. It's a great space.

Recall that we have defined a function

$$p : S^2 \to \mathbb{R}P^2$$

from the sphere to $\mathbb{R}P^2$. It sends a point $x \in S^2$ to the unique line L passing through x and the origin.



p is a surjection because every line through the origin passes through some point on the sphere.

p is not an injection. Indeed, every line through the origin passes through *two* points on the sphere. Thus p is a *two-to-one* map, meaning that every point in the codomain has a preimage of size two.

Remark 12.2.0.1. Fix a line $L \in \mathbb{R}P^2$. Note that if x is a point in $L \cap S^2$, then the point $-x$, defined by

$$x = (x_1, x_2, x_3) \implies -x = (-x_1, -x_2, -x_3)$$

is the other point in $L \cap S^2$. So we see that $p^{-1}(L) = \{x, -x\}$.

Definition 12.2.0.2 (Quotient topology, I. This is from homework.)**.** Let X be a topological space. If $p : X \to Y$ is a surjection, we topologize Y as follows: A subset $V \subset Y$ is open if and only if $p^{-1}(V)$ is open in X.

On the other hand, we have:

Definition 12.2.0.3 (Quotient topology, II.)**.** Let X be a topological space and $\sim$ an equivalence relation on X. Let $q : X \to X/\sim$ be the quotient map—i.e., $q(x) = [x]$. We topologize $X/\sim$ so that $V \subset X/\sim$ is open if and only if $q^{-1}(V)$ is open.

Remark 12.2.0.4. Note that the second definition is a special case of the first, because $q : X \to X/\sim$ is always a surjection.

Now we put the two definitions together. Let $p : X \to Y$ be a surjection. Recall from last week that p defines an equivalence relation $\sim$ on X given by $x \sim x' \iff p(x) = p(x')$, and that we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & Y \\ & {\scriptstyle q}\searrow \quad \nearrow{\scriptstyle \phi} & \\ & X/\sim\,. & \end{array}$$

Here, ϕ is a function given by $\phi([x]) = p(x)$. That the diagram is commutative means that $p = \phi \circ q$. Moreover, we also saw last week that ϕ is a bijection.

If we give $X/\sim$ the quotient topology (II), then there is a *unique* topology on Y so that ϕ is not only a bijection, but a homeomorphism. This is the topology for which $V \subset Y$ is open if and only if $\phi^{-1}(V)$ is open in $X/\sim$.

Definition 12.2.0.5 (Quotient topology, III.)**.** Let $p : X \to Y$ be a surjection, and let $\sim$ be the equivalence relation $x \sim x' \iff p(x) = p(x')$. Then we topologize Y so that $V \subset Y$ is open if and only if $\phi^{-1}(V)$ is open in $X/\sim$.

We leave the following for you to verify:

Proposition 12.2.0.6. The definitions I and III yield the same topology on Y.

Definition 12.2.0.7. Let X be a topological space and let $p : X \to Y$ be a surjection. The topology of Definition I (or III) is called the *quotient topology* on Y.

Definition 12.2.0.8. Consider the surjection $p : S^2 \to \mathbb{R}P^2$ discussed above. We topologize $\mathbb{R}P^2$ using the quotient topology.

Remark 12.2.0.9. So we are using two of our "how to make a new space" constructions to define $\mathbb{R}P^2$. First, note that we have topologized S^2 by the subspace topology—$\mathbb{R}^3$ has a standard topology, and we give S^2 the subspace topology. Second, we have used the quotient space construction.

12.3 An open subset

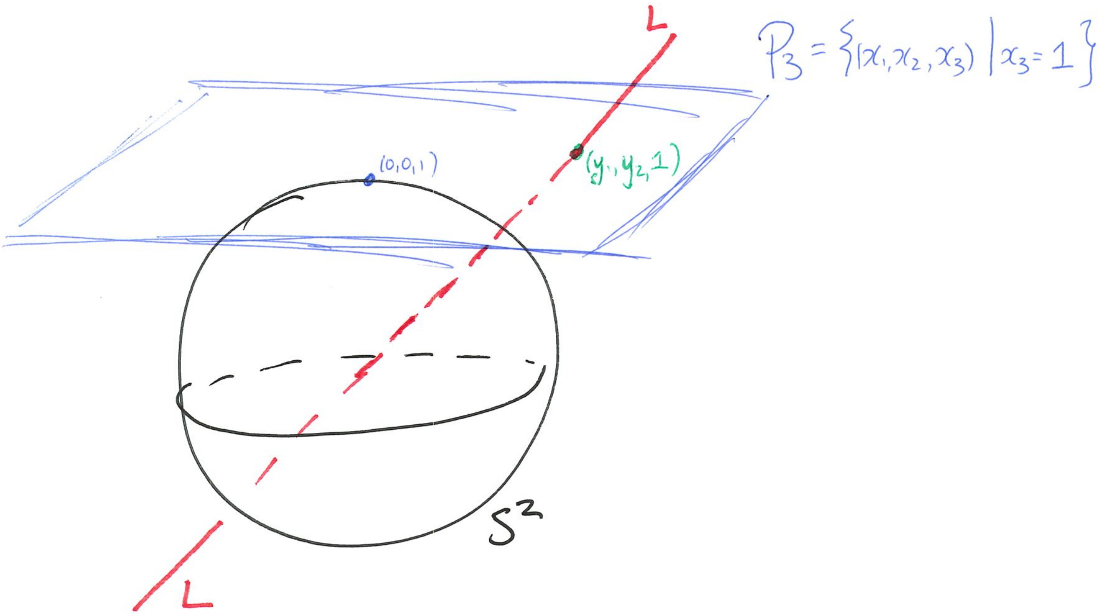
I would like to better understand $\mathbb{R}P^2$. So we're going to try to start understanding subsets of $\mathbb{R}P^2$ in terms of spaces I understand. Well, the only space that is remotely familiar to me is $\mathbb{R}^2$. So can I construct functions from $\mathbb{R}P^2$ to $\mathbb{R}^2$, and perhaps vice versa, that will help me understand $\mathbb{R}P^2$?

Here is a fun construction. Let $L \in \mathbb{R}P^2$ be a line through the origin. And fix a plane P_3 given by the equation $x_3 = 1$. Concretely,

$$P_3 = \{(x_1, x_2, x_3) \text{ such that } x_3 = 1\} \subset \mathbb{R}^3.$$

Then, if L intersects P_3, we have a unique intersection point $y \in L \cap P_3$. If we call its coordinates y_1, y_2, y_3, we know $y_3 = 1$, so we may as well only remember the pair (y_1, y_2). This yields an assignment

$$L \mapsto (y_1, y_2).$$



That is, it seems we almost have a function from $\mathbb{R}P^2$ to $\mathbb{R}^2$.

I say *almost* because not every line $L \in \mathbb{R}P^2$ intersections P_3. Indeed, what if L is the line given by the x_1 or the x_2 axis? In general, if L is *parallel* to the plane P_3, then L never intersects P_3, and we have no way of producing the numbers (y_1, y_2).

So this geometric construction doesn't produce a function from $\mathbb{R}P^2$ to $\mathbb{R}^2$, but it does produce a function from a *subset* of $\mathbb{R}P^2$ to $\mathbb{R}^2$. Let's give this subset a name.

Notation 12.3.0.1. Let $U_3 \subset \mathbb{R}P^2$ the set of those lines that intersect P_3.

Then the above construction defines a function

$$j_3 : U_3 \to \mathbb{R}^2, \qquad L \mapsto (y_1, y_2)$$

where $(y_1, y_2, 1)$ is the unique point in $L \cap P_3$.

Here is the big question of the day: Is U_3 open?

12.4 Proving U_3 is open

This is a great exercise in all the definitions.

12.4.1 Using the definition of quotient topology to reduce the problem to a subset of S^2

By definition, $U_3 \subset \mathbb{R}P^2$ is open if and only if its preimage in S^2 is open (its preimage under the map $p : S^2 \to \mathbb{R}P^2$).

Remember that p is the map sending a point x to the line passing through x and the origin. As such, the preimage of U_3 is the set of those $x \in S^2$ such that the line through x and the origin also passes through the plane P_3.

But given x, the line through x and the origin intersects P_3 if and only if the coordinate x_3 of x is non-zero. Thus, we find

$$p^{-1}(U_3) = \{x = (x_1, x_2, x_3) \in S^2 \text{ such that } x_3 \neq 0\}.$$

Let us call this set V_3.

Thus, to see whether U_3 is open, we must test whether V_3 is an open subset of S^2.

12.4.2 Using the definition of subset topology to reduce the problem to a subset of $\mathbb{R}^3$

Recall that S^2 is given the *subspace* topology as a subspace of $\mathbb{R}^3$. By definition, a subset $V \subset S^2$ is open if and only if there exists an open $W \subset RR^3$ for which

$$V = W \cap S^2.$$

Just as an ϵ-δ proof requires you to produce a δ given an ϵ, we must now exhibit a W given a V to prove that V is open.

Our V in question is the set $V_3 \subset S^2$ be the set of those $x \in S^2$ whose x_3 coordinate is non-zero. I claim V_3 is open.

So what is the open set $W \subset \mathbb{R}^3$?

Proposition 12.4.2.1. Let $W \subset RR^3$ denote the set of all elements $x \in \mathbb{R}^3$ for which $x_3 \neq 0$. Then :

(1) W is open in $\mathbb{R}^3$.

(2) $W \cap S^2 = V_3$.

Proof. We will omit the proof of (2). If (2) is not clear to you, just carefully think about the definitions.

To prove (1), it suffices to prove that any $x \in W$ is contained in some open ball $B(x,r)$ such that $B(x,r) \subset W$.[1]

Well, given $x \in W$, we know that x has distance $r = |x_3|$ from the plane $\{x_3 = 0\}$. (This is otherwise known as the x_1-x_2 plane.) Thus any element $x' \in \mathbb{R}^3$ of distance less than r is also contained in W. We conclude that $B(x, |x_3|) \subset W$, so W is open. □

Using the proposition, we conclude that $V_3 \subset S^2$ is an open subset. Because $V_3 = p^{-1}(U_3)$, we further conclude that $U_3 \subset \mathbb{R}P^2$ is an open subset.

12.5 Another open subset

So we have produced an open subset $U_3 \subset \mathbb{R}P^2$, and a function

$$j_3 : U_3 \to \mathbb{R}^2$$

which sends a line L intersecting the plane $P_3 = \{x_3 = 1\}$ to the the first two coordinates of the intersection point $L \cap P_3$.

Note that we did not need to choose the x_3 coordinate. For example, if we had chosen the x_2 coordinate, we could intersect lines with the plane $P_2 = \{x_2 = 1\}$. As before, we see that note every $L \in \mathbb{R}P^2$ intersects P_2; so let $U_2 \subset \mathbb{R}P^2$ be the set of those lines that intersect P_2.

We then have a function

$$j_2 : U_2 \to \mathbb{R}^2$$

given by sending a line L to the pair (y_1, y_3) where $(y_1, 1, y_3)$ is the intersection point of L with P_2. As before, we see that U_2 is an open subset of $\mathbb{R}P^2$.

12.6 A cover

Question: Do U_2 and U_3 cover $\mathbb{R}P^2$? That is,

$$\text{Does } U_2 \cup U_3 \text{ equal } \mathbb{R}P^2?$$

Parsing the definitions, we see that the union $U_2 \cup U_3$ consists of those lines L which pass through at least one of P_2 or P_3.

[1] We saw in a previous class that a subset W of a metric space is open if and only if for every $x \in W$, there is some open ball of positive radius containing x and contained in W.

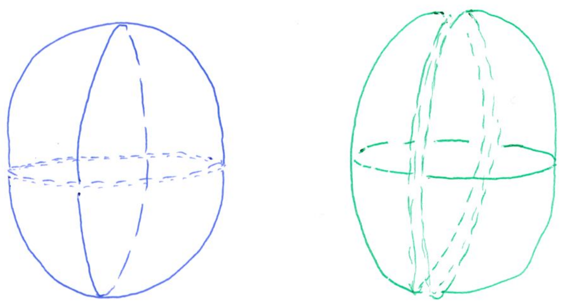


Figure 12.1: Open subsets V_3 (in blue) and V_2 (in green) of S^2.

Then the answer to the question is no. For example, if $U_2 \cup U_3$ were to equal $\mathbb{R}P^2$, then their preimages $V_2 = p^{-1}(U_2)$ and $V_3 = p^{-1}(U_3)$ would have the property that $V_2 \cup V_3 = S^2$, because p is a surjection. But indeed, $V_2 \cup V_3$ is missing exactly two points of the sphere: $(\pm 1, 0, 0)$.

To see this directly from the "set of lines" definitions, note that there is a line, called the x_1-axis, which does not pass through the plane P_2, nor the plane P_3. Indeed, this is the *only* line that does not pass through either of the planes. (Any other line would have a point with either the x_2 or x_3 coordinate being non-zero; in particular, such a line would intersect the plane P_2 (if $x_2 \neq 0$) or P_3 (if $x_3 \neq 0$).)

That is, $U_2 \cup U_3$ is equal to $\mathbb{R}P^2$ with one point removed.

But I want *all* of $\mathbb{R}P^2$.

Well, there is a notationally suggestive thing we can do: Let's define $U_1 \subset \mathbb{R}P^2$ to consist of those lines that pass through the plane $P_1 = \{x_1 = 1\}$. (This P_1 is the plane consisting of those vectors whose x_1 coordinate is equal to 1.) As before, we see that U_1 is open. It also clearly contains the x_1-axis. To summarize, we have:

Proposition 12.6.0.1. For $i = 0, 1$, or 2, let

$$P_i \subset \mathbb{R}^3$$

denote the set of those points whose x_ith coordinate is equal to 1. We let

$$U_i \subset \mathbb{R}P^2$$

consist of those lines L such that $L \cap P_i$ is non-empty. Then

1. each U_i is an open subset of $\mathbb{R}P^2$. Moreover,
2. The union
$$U_1 \cup U_2 \cup U_3$$
is equal to $\mathbb{R}P^2$.

We will study these open sets more next time.