Lecture 13

Understanding $\mathbb{R}P^2$ more

Most of today was spent going over Homework 6. But it's a long problem. Solutions are posted online.

Recall that last time, we defined three open subsets

$$U_1, U_2, U_3$$

of $\mathbb{R}P^2$.

Here, $U_i \subset \mathbb{R}P^2$ is the collection of all lines L such that L intersects the plane

$$P_i = \{x_i = 1\}.$$

(This notation is shorthand for the set of all points $x = (x_1, x_2, x_3)$ for which $x_i = 1$.)

We saw last time that each $U_i \subset \mathbb{R}P^2$ is open, and that $U_1 \cup U_2 \cup U_3 = \mathbb{R}P^2$.

13.1 Each U_i is a copy of $\mathbb{R}^2$

Now, suppose that $L \in U_3$. Then L intersects the plane P_3 (i.e., the plane of points whose 3rd coordinate is 1). So we can write

$$L \cap P_3 = \{(y_1, y_2, 1)\}$$

where $(y_1, y_2, 1)$ is the unique point in the intersection $L \cap P_3$. This defines for us a function

$$j_3 : U_3 \to \mathbb{R}^2, \qquad L \mapsto (y_1, y_2)$$

where (y_1, y_2) are the first two coordinates of the intersection point $L \cap P_3$.

It turns out that j_3 is continuous. Informally, this is because if we have some open ball around (y_1, y_2), then its preimage will consist of all lines that are close enough to L.

In fact, this is true for any of the U_i. More precisely, note that we have a function

$$j_1 : U_1 \to \mathbb{R}^2, \qquad L \mapsto (y_2, y_3)$$

where $(1, y_2, y_3)$ is the unique intersection point $L \cap P_1$. We also have a function

$$j_2 : U_2 \to \mathbb{R}^2, \qquad L \mapsto (y_1, y_3).$$

More is true:

Proposition 13.1.0.1. For every $i = 1, 2, 3$ the function

$$j_i : U_i \to \mathbb{R}^2$$

is a homeomorphism.

13.2 Paper mache

Now recall that homeomorphism is our notion of equivalence for topological spaces. So, informally, each time we see a U_i, we can replace it with $\mathbb{R}^2$.

On the other hand, we know that $\mathbb{R}P^2$ is a union of U_1, U_2 and U_3; because each U_i is homeomorphic to $\mathbb{R}^2$, this means that $\mathbb{R}P^2$ is actually a union of three spaces that are all copies of $\mathbb{R}^2$.

Informally, this means we can make $\mathbb{R}P^2$ out of some sort of paper mache—we put together three sheets of paper, but overlapping in some clever way, to make $\mathbb{R}P^2$.