Lecture 14

Atlases and transition functions for $\mathbb{R}P^2$

14.1 The earth is a sphere

I asked a question: How do you know that the earth's surface is (roughly) a sphere?

We were given many great ideas; one of the most convincing was to go into outer space and take a bunch of pictures. All others suffered from being based on taking local measurements, then assuming that some principle allowed us to conclude that those local measurements were valid anywhere on earth. The problem: We can't know that something about Point A on earth is true at Point B.

I claimed that the cheapest way to conclude that the earth is (roughly) a sphere is as follows: Go get an atlas of the earth. Rip out all the pages. Now glue the pages together along their overlaps. (For example, if Page 10 contains Lagos, and Page 33 does, too, then you should glue Page 10 and Page 33 together along where Lagos is displayed.) You are making a very complicated paper mache. And I claim that, in the end, you will end up with something that is roughly spherical.

(There are some issues: You have to assume that the atlas is correct. And indeed, to make a statement about something as large as the earth, you do need to rely on the accuracy of others' knowledge. Another issue is scaling; the scale of Page 10 may not equal the scale of Page 13; so you may have to find an atlas whose pages are made of rubber.)

If you take each page and remove its boundary edges, each page is homeomorphic to $\mathbb{R}^2$ (e.g., to an open rectangle). And each boundary-removed page is then an open subset of the surface of the earth (i.e., of the sphere). What you have just imagined is a procedure of finding a bunch of subsets of S^2 that are all homeomorphic to $\mathbb{R}^2$, and then writing S^2 as a union of these subsets. The way the subsets overlap tells you how to put together the paper mache.

14.2 Paper mache for $\mathbb{R}P^2$

So let's do this for $\mathbb{R}P^2$.

Recall from the last classes:

1. We define $U_1 \subset \mathbb{R}P^2$ to be the set of lines that intersect the plane $P_1 = \{x = (x_1, x_2, x_3) \text{ such that } x_1 = 1\}$. We have seen that this is an open subset, and I have told you that it is homeomorphic to $\mathbb{R}^2$.

   Likewise, we have open subsets U_2 and U_3 of $\mathbb{R}P^2$. Each of these is also homeomorphic to $\mathbb{R}P^2$. We have also seen that

   $$U_1 \cup U_2 \cup U_3 = \mathbb{R}P^2.$$

2. Moreover, the homeomorphisms from the U_i to $\mathbb{R}^2$ is given by functions

   $$j_i : U_i \to \mathbb{R}^2.$$

   Let us recall how these were defined. Given a line $L \in U_1$, so that L intersects the plane P_1, we can write the intersection point of L and P_1 as follows:

   $$(1, y_2, y_3).$$

   The function j_1 sends L to the pair of numbers (y_2, y_3).

   Likewise,

   $j_2(L) = (y_1, y_3)$ when $L \in U_2$, and $\qquad j_3(L) = (y_1, y_2)$ when $L \in U_3$.

14.2.1

Now onto some new material.

Because $U_1 \cup U_2 \cup U_3 = \mathbb{R}P^2$, we see that the induced map

$$h : U_1 \coprod U_2 \coprod U_3 \to \mathbb{R}P^2$$

is a surjection. (Note that the domain here is the coproduct of U_1, U_2, and U_3.) In particular, there exists an equivalence relation $\sim$ on $U_1 \coprod U_2 \coprod U_3$ such that we have an induced bijection

$$(U_1 \coprod U_2 \coprod U_3)/\sim \cong \mathbb{R}P^2.$$

This equivalence relation is one we've seen before: We declare

$$L \sim L' \iff h(L) = h(L').$$

(In general, when we have a surjection $h : X \to Y$, we can define a relation $x \sim x' \iff h(x) = h(x')$ so that we have an induced bijection $X/\sim \to Y$.)

Moreover, because each $U_i \subset \mathbb{R}P^2$ is open, we have:

Proposition 14.2.1.1. The induced map

$$(U_1 \coprod U_2 \coprod U_3)/\sim \to \mathbb{R}P^2$$

is a homeomorphism.

Now, because each $j_i : U_i \cong \mathbb{R}^2$ is a homeomorphism, we can begin to understand what $\mathbb{R}P^2$ looks like using coordinates on $\mathbb{R}^2$. For example, by using the three homeomorphisms, we have a single homeomorphism

$$j : U_1 \coprod U_2 \coprod U_3 \cong \mathbb{R}^2 \coprod \mathbb{R}^2 \coprod \mathbb{R}^2.$$

Thus we can try to understand the equivalence relation on the lefthandside in terms of the righthand side. That is, the homeomorphism j induces an equivalence relation on the right. What is this relation?

To see what it is, consider the functions

$$\begin{array}{ccccc} \mathbb{R}^2 & & & & \mathbb{R}^2 \\ \downarrow{\scriptstyle j_1^{-1}} & & & & \downarrow{\scriptstyle j_2^{-1}} \\ U_1 & \xrightarrow{\iota_1} & \mathbb{R}P^2 & \xleftarrow{\iota_2} & U_2. \end{array}$$

Then an element $y \in \mathbb{R}^2$ on the left is related to an element $y' \in \mathbb{R}^2$ on the right if and only if

$$\iota_1 \circ j_1^{-1}(y) = \iota_2 \circ j_2^{-1}(y').$$

I claim there is a formula now expressing y in terms of y'. To see this, note that the above equality means that $j_1^{-1}(y)$ and $j_2^{-1}(y')$ must describe the same line (i.e., the same element in $\mathbb{R}^2$). But j_1^{-1} takes the point $y = (y_2, y_3) \in \mathbb{R}^2$ and sends it to the line passing through

$$(1, y_2, y_3).$$

Likewise, j_2^{-1} takes the point $y' = (y_1', y_3') \in \mathbb{R}^2$ and sends it to the line passing through

$$(y_1', 1, y_3').$$

If these points are to be on the same line, then there must be a non-zero real number t so that

$$t(1, y_2, y_3) = (y_1', 1, y_3').$$

That is,

$$t = y_1', \qquad ty_2 = 1, \qquad ty_3 = y_3'.$$

From this, we quickly conclude that for the points $(1, y_2, y_3)$ and $(y_1', 1, y_3')$ to be on the same line L, we must have

$$y_2 = 1/y_1', \qquad y_3 = y_3'/y_1'.$$

Or, equivalently,

$$y_1' = 1/y_2, \qquad y_3' = y_3/y_2.$$

So this gives part of the relation; we see that y and y' are related if and only if the above equations hold. (In particular, y_2 must be non-zero for y to be related to some y', and y_1' must be non-zero for y' to be related to some y.)