Lecture 16

Closed sets and open covers

Class today will have three parts. As I mentioned last week, we're starting a proof bootcamp.

This means every day, you will see new definitions. Then you will spend most of class trying to prove something using the new definitions.

16.1 Closed sets

Definition 16.1.0.1. Let X be a topological space. A subset $A \subset X$ is called *closed* if and only if its complement is open.

I want you to prove the following in groups:

Proposition 16.1.0.2. Let $(X, \mathcal{T})$ be a topological space. For this problem, we let $\mathcal{K}$ denote the collection of closed subsets. Show the following are true:

1. $\emptyset, X \in \mathcal{K}$.

2. If $A_1, \ldots, A_n \in \mathcal{K}$ is a finite collection, then $\bigcup_{i=1,\ldots,n} A_i$ is in $\mathcal{K}$.

3. For an arbitrary collection $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ of elements of $\mathcal{K}$, we have that the intersection $\bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is also in $\mathcal{K}$.

Proposition 16.1.0.3. Let $f : X \to Y$ be a continuous map of topological spaces. Show that if $A \subset Y$ is closed, then its preimage is closed.

Conversely, suppose that $f : X \to Y$ is a function such that whenever $A \subset Y$ is closed, its preimage is closed. Prove that f is continuous.

Proposition 16.1.0.4. Let $B \subset X$ be an arbitrary subset. Show that there exists a subset, $\overline{B} \subset X$, satisfying the following properties:

1. $B \subset \overline{B}$
2. $\overline{B}$ is closed.
3. Moreover, if C is any other closed subset of X containing B, then C contains $\overline{B}$.

Informally, this means that $\overline{B}$ is the "smallest" closed subset of X containing B.

Definition 16.1.0.5 (For future use)**.** $\overline{B}$ is called the *closure* of B.

Remark 16.1.0.6 (Motivation for closed sets)**.** Note that the set of closed sets of a space can automatically recover the set of open sets of a space. (This is because $K \subset X$ is closed if and only if its complement is open.) If you expound upon Proposition 16.1.0.2, you will see that you can equivalently define a topological space through its closed sets, so long as the collection $\mathcal{K}$ of closed sets of X satisfy all the properties in the proposition. It is then an exercise to show that any such collection $\mathcal{K}$ determines a topology $\mathcal{T}$ (by taking the opens to be complements of elements of $\mathcal{K}$).

The next proposition tells you that you can also equivalently define the notion of continuity through mentioning only closed sets.

One thing you can do freely with closed sets is take intersections, as you saw in Proposition 16.1.0.2. This allows you to do constructions with closed sets you can't do with open sets. For example, one can convert any set into a closed by taking its closure (which you saw in Proposition 16.1.0.4). This, informally, gives you a "slightly larger" but closed subset.

In contrast, given some subset $B \subset X$ of a topological space, it is almost impossible to construct a "smallest open set" containing B; you can rather construct the "largest open set" contained in B, and this is called the interior of B. Can you construct it?

16.2 Open covers

Definition 16.2.0.1. Let $(X, \mathcal{T}_X)$ be a topological space. We say that a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets of X is a *cover* if $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = X$. We further say this collection is an *open cover* if each U_α is open.

I want you to prove the following:

Proposition 16.2.0.2. Let $\{U_\alpha\}$ be an open cover of X. Note there is a function

$$p : \coprod_{\alpha \in \mathcal{A}} U_\alpha \to X.$$

Prove that the induced map

$$(\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \to X$$

is a homeomorphism. (Here, the equivalence relation $\sim$ is the one for which $x \sim x' \iff p(x) = p(x')$.

Remark 16.2.0.3 (Motivation for open covers)**.** Proposition 16.2.0.2 says that you can reconstruct a space X from an open cover of X. This is something special to *open* covers.

For example, given an arbitrary cover $\{A_\alpha\}_{\alpha \in \mathcal{A}}$, even if $\bigcup_{\alpha \in \mathcal{A}} A_\alpha = X$, it need not be true that

$$\left(\coprod_{\alpha \in \mathcal{A}} A_\alpha\right)/\sim \to X$$

is a homeomorphism. For example, you could take $\mathcal{A} = X$ and $A_x = \{x\}$. Then the above map is a homeomorphism if and only if X has the discrete topology.

16.3 The Mobius band in $\mathbb{R}P^2$

In the exam I asked you to convince me there is a Möbius band inside the union $U_1 \cup U_2 \subset \mathbb{R}P^2$.

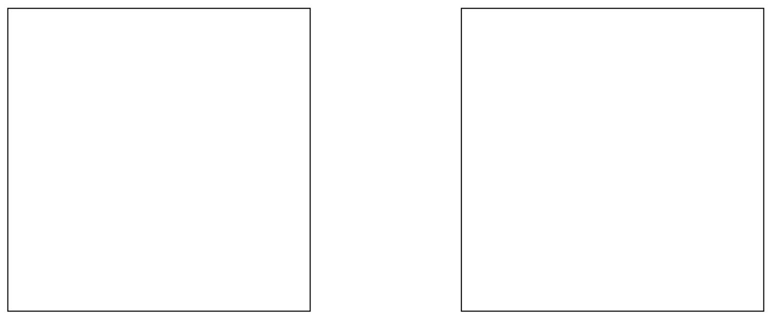
Recall that $U_1 \cup U_2$ is homeomorphic to the following:

$$(\mathbb{R}^2 \coprod \mathbb{R}^2)/\sim$$

where the relation $\sim$ says $y \sim y' \iff$

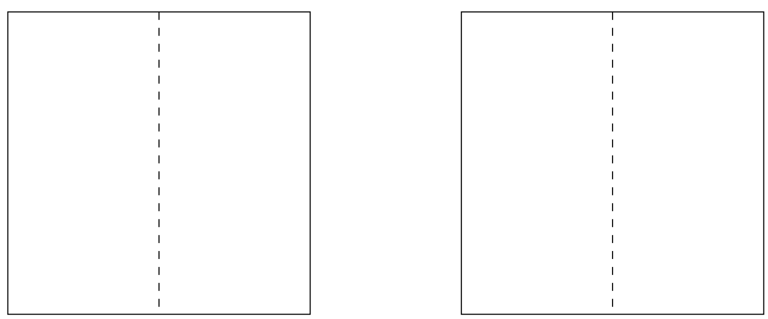
$$\begin{cases} y_2 = 1/y'_1 \text{ and } y_3 = y'_3/y_1, & y \text{ is in the first copy of } \mathbb{R}^2 \text{ while } y' \text{ is in the second copy.} \\ y = y' & y \text{ and } y' \text{ are both in the first copy of } \mathbb{R}^2 \\ y' = y & y \text{ and } y' \text{ are both in the second copy of } \mathbb{R}^2 \\ y'_2 = 1/y_1 \text{ and } y'_3 = y_3/y_1, & y \text{ is in the second copy of } \mathbb{R}^2 \text{ while } y' \text{ is in the first copy.} \end{cases}$$

The relation looks more complicated than it needs to; if you are willing, you can simply think of $\sim$ as the smallest equivalent relation possible containing the first line above.



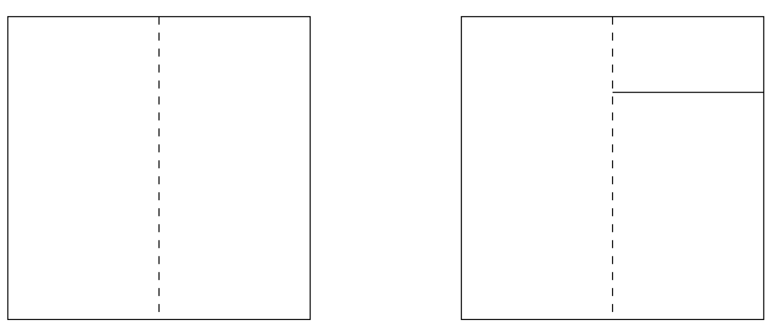
$\mathbb{R}^2 = \{(y_2, y_3)\}$ $\mathbb{R}^2 = \{(y'_1, y'_3)\}$

Above is a picture of two copies of $\mathbb{R}^2$.



$\mathbb{R}^2$ $\mathbb{R}^2$

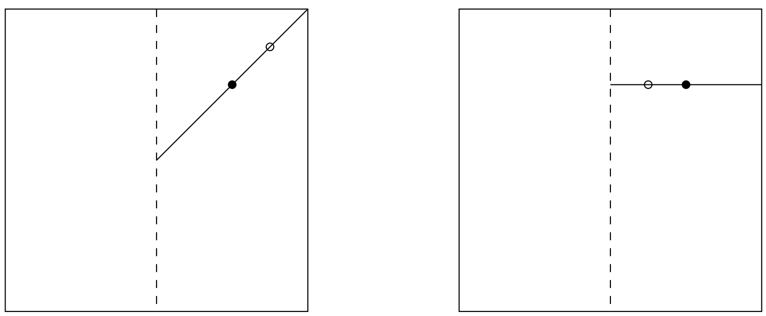
We have drawn (dashed) the lines $y_2 = 0$ (in the lefthand copy of $\mathbb{R}^2$) and the $y'_1 = 0$ (in the righthand copy of $\mathbb{R}^2$). I draw these because these points are only related to themselves; they undergo no gluing.



On the right we have drawn (a portion of) the horizontal line $y_3' = a$ for some positive real number a. We have only drawn the portion where $y_1' > 0$. What points on the right are points on $\{y_3' = a\}$ related to? Well, we know

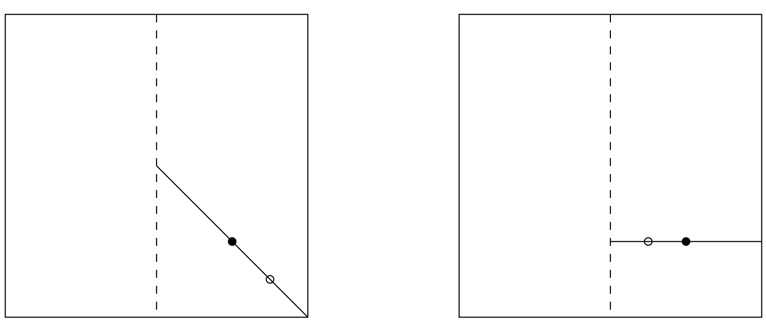
$$y \sim y' \iff y_2 = 1/y_1', y_3 = y_3'/y_1'.$$

In other words, a point of the form $(y_1', y_3') = (y_1', 1)$ on the right is related to a point of the form $(y_2, y_3) = (1/y_1', 1/y_1')$ on the left. These are points where the y_2 and y_3 coordinates are equal; i.e., this is some part of a line! Let's draw the portion where $y_1' > 0$:

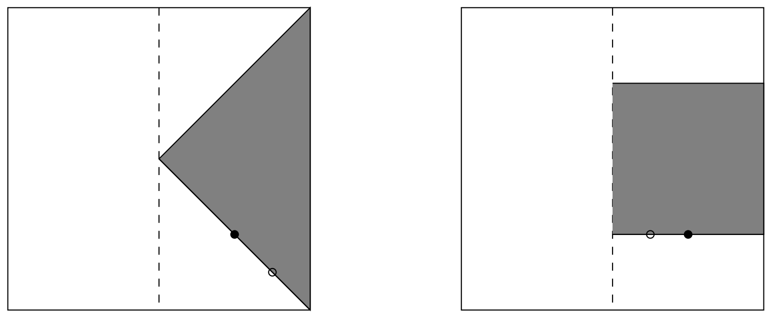


Note, importantly (see the white dot) that when the y_1' coordinate shrinks toward 0, the y_2 coordinate on the right *increases*.

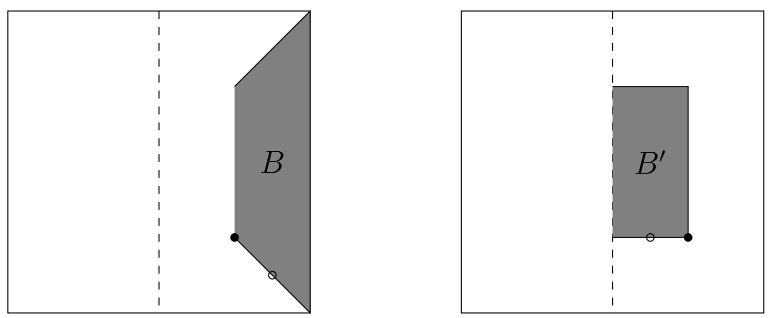
We can likewise draw how the ray $y_3' = -1$, with y_1' positive, is related to a ray in the lefthand side, by reasoning that $(y_1', -1)$ on the right is related to $(1/y_1', -1/y_1')$ on the left. (Thus, points on this horizontal ray on the right, are related to points on a line of slope -1 on the left.)



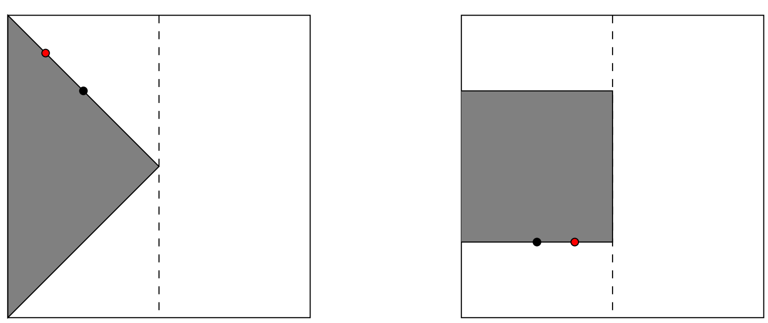
All told, we see that the shaded regions are related to each other as follows:



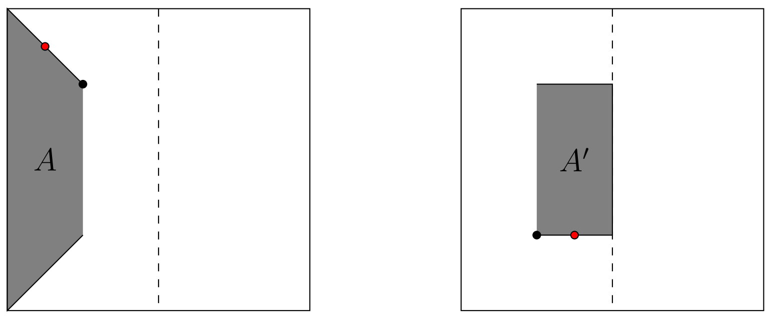
For reasons that will become clear later, let's just remember the shaded region B' on the right, and the shaded region B on the left:



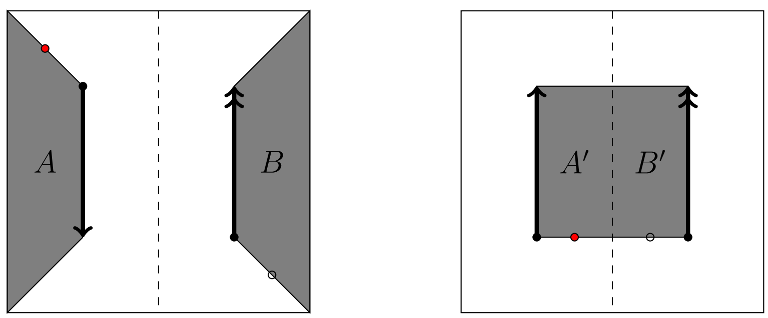
Now I leave it to you to explore what happens when the y'_1 coordinate on the righthand copy of $\mathbb{R}^2$ is *negative*, which we haven't considered yet. I claim you'll get the following picture (where points on the shaded region are related to each other in a way I want you to figure out):



Note importantly that though the y'_3 coordinates of the dots on the righthand side were negative, the y_3 coordinate of the related points on the righthand side are *positive*! Now consider the regions A and A' indicated below:

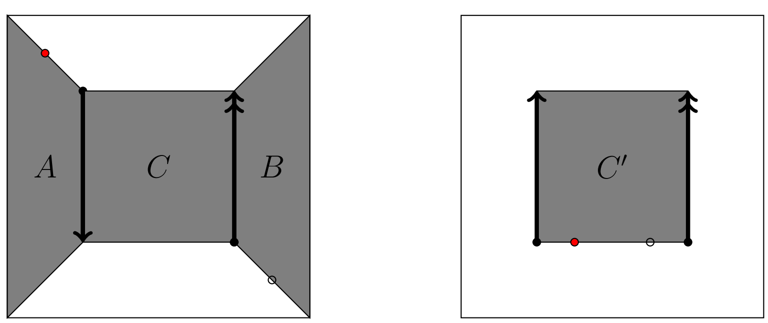


For your convenience, let me re-draw the regions $A, B \subset \mathbb{R}^2$ and $A', B' \subset \mathbb{R}^2$:



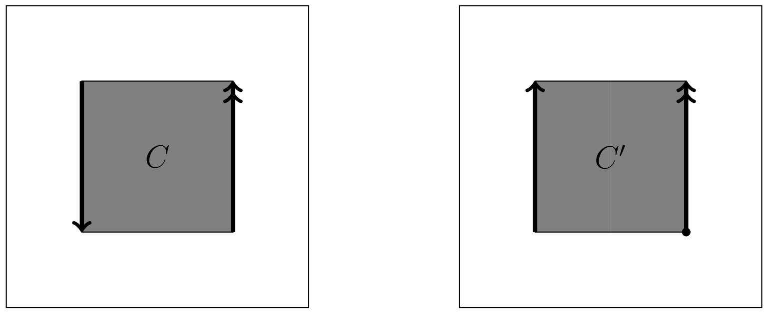
I want to emphasize that A' and B' do not touch; they share no intersection! (The dashed line is important.)

So finally I am ready to draw the Mobius band inside of $U_1 \cup U_2$. Consider the regions C and C' below:



For example, C' contains both A' and B', and a little bit more–it contains some points with $y_1' = 0$, for example.

While C is a subset of (the left copy of) $\mathbb{R}^2$, and C' is a subset of (the right copy of) $\mathbb{R}^2$, they are mapped to a $U_1 \cup U_2$ in a way such that they overlap along the arrowed edges. The overlap is interesting; as indicated, the left edge of C' is glued to the left edge of C in a way that "flips" orientation.



Now I leave it to you to glue C and C' together along the edges, as indicated; you wil get a Mobius strip.

Proof of Propositions

Proof of Proposition 16.1.0.2. Note that in this problem, given $A \subset X$, the complement of A is the complement of A in X. That is, if A^C denotes the complement, we have

$$A^C = \{x \in X \text{ such that } x \notin A.\}.$$

1. To show the empty set is closed, we must show its complement is open. We know $\emptyset^C = X$, and by the definition of topological space, we know X is open. This shows that the empty set is closed.

To show that X is closed, we must show that its complement is open. We know $X^C = \emptyset$, and the empty set is always open (by definition of topological space). Thsi shows that X is closed.

2. Since each A_i is closed, we know A_i^C is open for every $i = 1, \ldots, n$. By DeMorgan's Laws, we have

$$(\bigcup_{i=1,\ldots,n} A_i)^C = \bigcap_{i=1,\ldots,n} (A_i^C).$$

The righthand side is a finite intersection open sets. Hence it is open (by definition of topological space). Because $(\bigcup_{i=1,\ldots,n} A_i)^C$ is open, we conclude that $\bigcup_{i=1,\ldots,n} A_i$ is closed (by definition of closed set). This finishes the proof.

3. Again by DeMorgan's Laws, we have

$$(\bigcap_{\alpha\in\mathcal{A}} A_\alpha)^C = \bigcup_{\alpha\in\mathcal{A}} (A_\alpha^C).$$

Each A_α^C is open because each A_α is closed; thus the righthand side is a union of open sets. Thus the righthand side is open (by definition of topological space). This shows that $(\bigcap_{\alpha\in\mathcal{A}} A_\alpha)^C$ is open, which means $\bigcap_{\alpha\in\mathcal{A}} A_\alpha$ is closed (by definition of closed set). □

Proof of Proposition 16.1.0.3. Let $A \subset Y$ be closed. Then A^C is open. Moreover,

$$f^{-1}(A^C) = f^{-1}(A)^C.$$

(To see this, you need to exhibit each set as a subset of the other. Well, x is in the lefthand side if $f(x) \notin A$. In particular, $x \notin f^{-1}(A)$. Likewise, if x is in the righthand side, then $x \notin f^{-1}(A)$, so $f(x) \notin A$, meaning $x \in f^{-1}(A^C)$.)

And the lefthand side is open by definition of continuous map. Thus $f^{-1}(A)^C$ is open, meaning $f^{-1}(A)$ is closed. □

Proof of Proposition 16.1.0.4. Omitted until next time. □

Proof of Proposition 16.2.0.2. Let us recall the definition of the disjoint union $\coprod_{\alpha,\mathcal{A}} U_\alpha$—this is the set of all pairs (x,α) where $\alpha \in \mathcal{A}$ and $x \in U_\alpha$. Then the function p is given by

$$p : \coprod_{\alpha\in\mathcal{A}} U_\alpha \to X, \qquad (x,\alpha) \mapsto x.$$

That is, $p(x,\alpha) = x$.

Because $\{U_\alpha\}$ is a cover of X, for every $x \in X$, there is some α such that $x \in U_\alpha$. In particular, for every $x \in X$, there is some (x,α) such that $p(x) = x$. This shows that p is a surjection.

In a previous lecture, we showed that whenever p is a surjection, then the induced function

$$\text{domain of } p/\sim \to \text{ codomain of } p$$

is a bijection if $\sim$ is defined by $x \sim x' \iff p(x) = p(x')$. This is exactly the equivalence relation that we are taking, so we conclude that the induced map

$$f : (\coprod_{\alpha\in\mathcal{A}} U_\alpha)/\sim \to X$$

is a bijection. Note that we have now given this map a name: f.

First let's show f is continuous. In homework you showed that f is continuous if and only if p is. (A map from the quotient is continuous if and only if the its composition with the projection map is.) So let us show p is continuous. This means we must show that if $V \subset X$ is an open subset, then $p^{-1}(V)$ is open. By definition of coproduct topology, $p^{-1}(V)$ is open if and only if

$$p^{-1}(V) \cap U_\alpha$$

is open for every $\alpha \in \mathcal{A}$. So let's prove it. First, we compute:

$$\begin{aligned} p^{-1}(V) \cap U_\alpha &= \{\ x \in U_\alpha \text{ such that } p(x) \in V\} && (16.3.1)\\ &= \{\ x \in U_\alpha \text{ such that } x \in V && (16.3.2)\\ &= V \cap U_\alpha. && (16.3.3) \end{aligned}$$

Because $V \subset X$ is open and $U_\alpha \subset X$ is open, their (finite) intersection is open. This shows that $p^{-1}(V) \cap U_\alpha$ is open for every $\alpha \in \mathcal{A}$, and hence that $p^{-1}(V)$ is open. This shows p is continuous. Thus f is continuous.

Now we must show that the inverse map

$$g : X \to (\coprod_\alpha U_\alpha)/\sim$$

is continuous. For this, it suffices to show that if a subset V in the codomain is open, then $g^{-1}(V) = f(V)$ is open. Well, something in the codomain is open if and only if its preimage under the quotient map is open (by definition of quotient topology). Thus, a subset V of the codomain is open if and only if V is the image of some open subset

$$\tilde{V} \subset \coprod_\alpha U_\alpha.$$

Again by definition of coproduct topology, $\tilde{V}$ then has the property that $\tilde{V} \cup U_\alpha$ is open for every α. But then we have

$$f(V) = p(\tilde{V}) = p(\bigcup_\alpha V \cap U_\alpha) = \bigcup_\alpha p(V \cap U_\alpha).$$

For every α, we know $V \cap U_\alpha$ is open an open subset of X (because it is a finite intersection of open sets), so $p(V \cap U_\alpha) = V \cap U_\alpha$ is an open subset of X. In other words, the rightmost term in the above string of equalities is a union of open sets, and is hence open (by definition of topology). This shows $f(V)$ is open, which completes the proof. $\square$