Lecture 17

Open covers and subcovers

Last time you began thinking about open covers. Let me remind you:

Definition 17.0.0.1 (Definition 16.2.0.1)**.** Let $(X, \mathcal{T}_X)$ be a topological space, and fix a set $\mathcal{A}$. Fix a function $\mathcal{U} : \mathcal{A} \to \mathcal{T}_X$. For every $\alpha \in \mathcal{A}$, we will write U_α for the value of this function on α. We say that $\mathcal{U}$ is an *open cover* if

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = X.$$

Remark 17.0.0.2. Note that it seems I have changed the definition!

Exercise 17.0.0.3 (Do only if you or your group decides to.)**.** Show that this definition is equivalent to the old one: " We say that a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets of X is a *cover* if $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = X$. We further say this collection is an *open cover* if each U_α is open."

17.1 Subcovers

Definition 17.1.0.1. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X.[1] A collection $\{U_\beta\}_{\beta \in \mathcal{B}}$ is called a *subcover* of $\mathcal{U}$ if

1. $\bigcup_{\beta \in \mathcal{B}} U_\beta = X$, and
2. For every $\beta \in \mathcal{B}$, there exists $\alpha \in \mathcal{A}$ such that $U_\beta = U_\alpha$.

[1] Note that I am not explicitly saying that X is a topological space here; it is to be inferred from context, because I am talking about an "open" cover of X—this only makes sense if I know what the open sets of X are!

Exercise 17.1.0.2 (Do only if you or your group decides to.)**.** Let $\mathcal{U}$ be an open cover. Then a subcover of $\mathcal{U}$ is the same data as a choice of subset $\mathcal{B} \subset \mathcal{A}$ such that the composition

$$\mathcal{B} \to \mathcal{A} \to \mathcal{T}$$

is an open cover of X.

Prove the following:

Proposition 17.1.0.3. Let $\mathcal{A} = X \times \mathbb{R}_{>0}$ be the set of pairs (x, r) where $x \in X$ and r is a positive real number. Let (X, d) be a metric space, and equip it with the induced topology.

(i) Show that the collection

$$\mathcal{U} = \{\text{Ball}(x, r)\}_{(x,r)\in\mathcal{A}}$$

is an open cover of X.

(ii) Now let $\mathcal{B} \subset \mathcal{A}$ denote the set of pairs (x, r) where $x \in x$ and r is a positive *rational* number. (So $\mathcal{B} = X \times \mathbb{Q}$.) Show that $\{U_\beta\}_{\beta\in\mathcal{B}}$ is a subcover of $\mathcal{U}$.

17.2 From last time

If you didn't have a chance last time, I want you to tackle the following:

17.2.1 Preimages of closed sets are closed

Proposition 17.2.1.1 (Proposition 16.1.0.3.)**.** Let $f : X \to Y$ be a continuous map of topological spaces. Show that if $A \subset Y$ is closed, then its preimage is closed.

Conversely, suppose that $f : X \to Y$ is a function such that whenever $A \subset Y$ is closed, its preimage is closed. Prove that f is continuous.

17.2.2 Open covers can reconstruct the space

Proposition 17.2.2.1 (Proposition 16.2.0.2.)**.** Let $\{U_\alpha\}$ be an open cover of X. Note there is a function

$$p : \coprod_{\alpha\in\mathcal{A}} U_\alpha \to X.$$

Prove that the induced map

$$(\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \;\to X$$

is a homeomorphism. (Here, the equivalence relation $\sim$ is the one for which $x \sim x' \iff p(x) = p(x')$.

17.2.3 Closures

Proposition 17.2.3.1 (Proposition 16.1.0.4.)**.** Let $B \subset X$ be an arbitrary subset. Show that there exists a subset, $\overline{B} \subset X$, satisfying the following properties:

1. $B \subset \overline{B}$
2. $\overline{B}$ is closed.
3. Moreover, if C is any other closed subset of X containing B, then C contains $\overline{B}$.

Informally, this means that $\overline{B}$ is the "smallest" closed subset of X containing B.

17.2.4 A Mobius band in $\mathbb{R}P^2$

Convince yourselves that there is a Möbius band inside the union $U_1 \cup U_2 \subset \mathbb{R}P^2$.