Lecture 18

Solutions to polynomial equations are closed

18.0.1 Open covers can reconstruct the space

If you haven't completed the proof of this proposition, I want you to keep working on it. It will give you practice with coproducts, quotients, the quotient topology, and homeomorphisms:

Proposition 18.0.1.1 (Proposition 16.2.0.2.)**.** Let $\{U_\alpha\}$ be an open cover of X. Note there is a function

$$p : \coprod_{\alpha \in \mathcal{A}} U_\alpha \to X.$$

Then the induced map

$$(\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \to X$$

is a homeomorphism. (Here, the equivalence relation $\sim$ is the one for which $x \sim x' \iff p(x) = p(x')$).

18.1 Familiar (?) examples of continuous functions

Going forward, you may rely on the following:

Exercise 18.1.0.1 (Do only if you want to.)**.** Show that addition,

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x_1, x_2) \mapsto x_1 + x_2$$

is continuous. (Here, $\mathbb{R}$ is given the topology induced by the standard metric.)

Exercise 18.1.0.2 (Do only if you want to.)**.** Show that the multiplication function

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x_1, x_2) \mapsto x_1 x_2$$

is continuous. (Here, $\mathbb{R}$ is given the topology induced by the standard metric.)

Exercise 18.1.0.3 (Do only if you want to.)**.** Show that the following functions are continuous:

1. Fix a real number $a \in \mathbb{R}$. The constant function

$$\mathbb{R} \to \mathbb{R}, \qquad x \mapsto a.$$

2. Fix two continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. The function

$$\mathbb{R} \to \mathbb{R} \times \mathbb{R}, \qquad x \mapsto (f(x), g(x)).$$

18.2 Polynomial functions are continuous

Exercise 18.2.0.1 (Do only if you want to.)**.** (You will need to rely on the exercises above. If you want, you can try proving the following propositions *without* proving the exercises yourself, but taking their truth for granted.)

1. Any polynomial function in one variable is continuous. That is, if one has a finite collection of real numbers $a_0, \ldots, a_n$, the function

$$p : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n = \sum_{i=0}^{n} a_i x^i$$

is continuous. (Hint: Induction on n.)

2. Any polynomial function in finitely many variables is continuous. That is, if we are given a real number $a_{i_1,\ldots,i_m}$ for some finite collection of m-tuples of non-negative integers $i_1, \ldots, i_m$, the function

$$\mathbb{R}^m \to \mathbb{R}, \qquad (x_1, \ldots, x_m) \mapsto \sum_{i_1,\ldots,i_m} a_{i_1,\ldots,i_m} x_1^{i_1} \ldots x_m^{i_m}$$

is continuous. (Hint: A lot of induction.)

18.3 Some closed subsets of $\mathbb{R}^n$

Prove the following:

Proposition 18.3.0.1.

1. Fix a real number $b \in \mathbb{R}$. Then the (singleton) set $\{b\} \subset \mathbb{R}$ is closed.

2. For every $m \geq 1$, the $(m-1)$-dimensional sphere

$$S^{m-1} \subset \mathbb{R}^m$$

is a closed subset of $\mathbb{R}^m$. (Recall that

$$S^{m-1} := \{(x_1, \ldots, x_m) \text{ such that } \sum_{i=1}^{m} x_i^2 = 1\}.$$

As a hint, you can use the fact that for continuous functions, preimages of closed subsets are closed.)

3. More generally, given any polynomial p in m variables, the set

$$\{x \text{ such that } p(x) = 0\} \subset \mathbb{R}^m$$

is a closed subset.

4. Even more generally, given a finite collection of polynomials $p_1, \ldots, p_k$ in m variables, the set

$$\{x \text{ such that } p_i(x) = 0 \text{ for all } i\} \subset \mathbb{R}^m$$

is a closed subset.

5. Even more generally, given an arbitrary collection of polynomials $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ in m variables, the set

$$\{x \text{ such that } p_\alpha(x) = 0 \text{ for every } \alpha \in \mathcal{A}\} \subset \mathbb{R}^m$$

is a closed subset.

Prove the following:

Proposition 18.3.0.2.

1. Fix a real number a. Then the set

$$(-\infty, a] \subset \mathbb{R}$$

is closed (under the standard topology).

2. Fix a real number a and let $p : \mathbb{R}^m \to \mathbb{R}$ be a polynomial function in m variables. Then the set

$$\{x \in \mathbb{R}^m \text{ such that } p(x) \leq a \}$$

is closed. If you need to, do the same for $\geq a$ rather than $\leq a$.

18.4 The Heine-Borel Theorem

If you have gotten this far, you can go onto facts that will be useful and that we will cover later.

Definition 18.4.0.1. A subset $A \subset \mathbb{R}$ is called *bounded* if there exists some positive real number $a \in \mathbb{R}$ for which

$$A \subset (-a, a).$$

More generally, given a subset $A \subset \mathbb{R}^n$, we say that A is bounded if there exists some positive real number $a \in \mathbb{R}$ for which

$$A \subset \text{Ball}(0; a).$$

Prove:

Theorem 18.4.0.2 (Heine-Borel Theorem)**.** A subset $A \subset \mathbb{R}^n$ is compact if and only if it is both closed and bounded.