Lecture 19

Closed balls in metric spaces; Heine-Borel

19.1 The metric function is continuous

Proposition 19.1.0.1. Let $d : X \times X \to \mathbb{R}$ be a metric. Endow X with the metric topology (i.e., the topology induced by the metric) and endow $X \times X$ with the product topology. $\mathbb{R}$ has the standard topology.

1. Show that d is continuous.

2. For any $x_0 \in X$, show that the function

$$d(x_0, -) : X \to \mathbb{R}, \qquad x \mapsto d(x_0, x)$$

is continuous.

Proposition 19.1.0.2. Let $d : X \times X \to \mathbb{R}$ be a metric. Endow X with the metric topology (i.e., the topology induced by the metric) and endow $X \times X$ with the product topology.

1. Fix a real number $a \in \mathbb{R}$. For every $x_0 \in X$, show that

$$\{ \ x \in X \text{ such that } d(x_0, x) = a \ \}$$

is a closed subset of X.

2. Fix a real number $a \in \mathbb{R}$. For every $x_0 \in X$, show that
$$\{ \ x \in X \text{ such that } d(x_0, x) \leq a \ \}$$
is a closed subset of X. This is called the *closed ball of radius* a *centered at* x_0.

19.2 Proving Heine-Borel

Definition 19.2.0.1 (Definition 18.4.0.1)**.** A subset $A \subset \mathbb{R}$ is called *bounded* if there exists some positive real number $a \in \mathbb{R}$ for which
$$A \subset (-a, a).$$
More generally, given a subset $A \subset \mathbb{R}^n$, we say that A is bounded if there exists some positive real number $a \in \mathbb{R}$ for which
$$A \subset \text{Ball}(0; a).$$

Prove:

Theorem 19.2.0.2 (Heine-Borel Theorem, 18.4.0.2.)**.** A subset $A \subset \mathbb{R}^n$ is compact if and only if it is both closed and bounded.

19.3 Summary of this week's knowledge

19.3.1 Examples of closed subsets, and polynomials

1. Polynomials are continuous functions
2. Solutions to polynomials are closed subsets
3. Solutions to inequalities defined by polynomials are closed subsets
4. In a metric space, closed balls are closed subsets

19.3.2 Closed and bounded subsets

1. You should know the Heine-Borel theorem, even if you don't know its proof.
2. You should be able to give examples of closed and bounded subsets of $\mathbb{R}^n$.