Lecture 20

Proving the Heine-Borel Theorem

Recall the following definitions:

Definition 20.0.0.1. A topological space is called *compact* if every open cover of the space admits a finite subcover.

Definition 20.0.0.2. A subset of a topological space X is called *closed* if its complement is open.

Definition 20.0.0.3. A subset A of $\mathbb{R}^n$ is called *bounded* if there is some $r \in \mathbb{R}$ such that

$$A \subset \mathrm{Ball}(0; r).$$

(That is, A is contained in a ball of radius r centered at the origin.)

Today, we will prove:

Theorem 20.0.0.4 (Heine-Borel theorem)**.** Fix $A \subset \mathbb{R}^n$. Then A is compact if and only if it is closed and bounded.

We will see pay-offs next class.

20.1 Just take these for granted

Here are a few lemmas. You should take them for granted; no need to prove them. Just read them and try to understand them.

Lemma 20.1.0.1. Fix two real numbers a, b such that $a \leq b$. Then the interval $[a, b]$ is compact. (We endow $[a, b] \subset \mathbb{R}$ with the subspace topology.)

Lemma 20.1.0.2. Let X and Y be compact. Then $X \times Y$ is compact.

Lemma 20.1.0.3 (You proved this in homework)**.** Let X and Y be topological spaces.

1. Let X be compact. Then any closed subset $A \subset X$ is compact.

2. Let Y be Hausdorff. Then any compact subset $B \subset Y$ is closed.

20.2 A proof of Heine-Borel

In your groups, read the following proof of the Heine-Borel theorem. Speak out when you do not understand some portion of the proof. Make sure you understand every step.

Proof. Fix n.

(Compact $\implies$ closed and bounded.) To begin, define a collection of open balls as follows:

$$W_r = \text{Ball}(0; r) \subset \mathbb{R}^n, \qquad r > 0.$$

Note that the collection $\{W_r\}_{r>0}$ forms an open cover of $\mathbb{R}^n$.

Now let $A \subset \mathbb{R}^n$ be compact, and define

$$U_r = W_r \cap A.$$

Then the collection $\{U_r\}$ forms an open cover of A. By compactness of A, there is a finite subcover, meaning there is a finite collection $r_1, \ldots, r_n$ such that

$$\bigcup_{i \in 1, \ldots, n} U_{r_i} = A.$$

But if $r > r'$, clearly $U_r \supset U_{r'}$, so letting $R = \max\{r_1, \ldots, r_n\}$, we have that $A \subset W_R$. This shows A is bounded.

To show A is closed, we simply cite Lemma 20.1.0.3(2). (Note that $\mathbb{R}^n$ is Hausdorff because it is a metric space.)

(Closed and bounded $\implies$ compact.) Now suppose $A \subset \mathbb{R}^n$ is closed and bounded. Well, because A is bounded, there is some real number r so that $A \subset \text{Ball}(0; r)$. In particular, A is contained in the square

$$[-r, r] \times \ldots \times [-r, r] \subset \mathbb{R}^n.$$

Here, the lefthand side consists of those points

$$(x_1, \ldots, x_n)$$

such that $x_i \in [-r, r]$ for all i. But the interval $[-r, r]$ is compact by Lemma 20.1.0.1; so by Lemma 20.1.0.2, the direct product

$$[-r, r] \times \ldots \times [-r, r]$$

is also compact. Moreover, we have

$$A \subset [-r, r] \times \ldots \times [-r, r] \subset \mathbb{R}^n$$

and because A is closed in $\mathbb{R}^n$, we know that A is closed in $[-r, r] \times \ldots [-r, r]$. Invoking Lemma 20.1.0.3(1), we conclude that A is compact. □

Just to make sure you understood every element of the proof:

1. We used induction at some point. Where?

2. At some point we had to verify that the subspace topology of $[-r, r] \times \ldots \times [-r, r] \subset \mathbb{R}^n$ equals the product topology of $[-r, r] \times \ldots \times [-r, r]$. Where was the first point we needed to do this?

20.3 Another proof to verify

Remark 20.3.0.1. We will not give a Proof of Lemma 20.1.0.1; it is proven in most analysis classes.

Make sure you understand every step. This proof should make for good group discussion.

Proof of Lemma 20.1.0.2. (I) Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of $X \times Y$. Before we treat a more general case, let us assume that every element of $\mathcal{U}$ is of the form $V_\alpha \times W_\alpha$, where $V_\alpha \subset X$ is open and $W_\alpha \subset Y$ is open.

For every $x \in X$, let us consider the subset $\{x\} \times Y$. By definition of cover, for every element $(x, y) \in \{a\} \times Y$, there is some α such that $(x, y) \in V_\alpha \times W_\alpha$. So choose some collection $\mathcal{A}_x \subset \mathcal{A}$ so that

$$\bigcup_{\alpha \in \mathcal{A}_x} V_\alpha \times W_\alpha \supset \{x\} \times Y.$$

Then the collection $\{W_\alpha\}_{\alpha \in \mathcal{A}_x}$ is an open cover of Y. Because Y is compact, there is some finite subcover—i.e., some finite subset $F_x \subset \mathcal{A}_x$ so that

$$\bigcup_{\alpha \in F_x} W_\alpha = Y.$$

Now consider the finite intersection

$$\bigcap_{\alpha \in F_x} V_\alpha.$$

This is an open subset of X, and we call it V_x. Note that $x \in V_x$, and

$$\bigcup_{\alpha \in F_x} V_x \times W_\alpha \supset \{x\} \times Y. \tag{20.3.0.1}$$

In this way, for every $x \in X$, we obtain an open subset $V_x \subset X$ such that $x \in V_x$, and such that there exists some finite subset $F_x \subset \mathcal{A}$ for which (20.3.0.1) holds.

The collection $\{V_x\}_{x \in X}$ forms an open cover of X. By compactness of X, there exists a finite subcover. Hence there is some finite collection of points $x_1, \ldots, x_n \in X$ so that

$$V_{x_1} \cup \ldots \cup V_{x_n} = X.$$

It follows that

$$\bigcup_{x_1, \ldots, x_n} \bigcup_{\alpha \in F_{x_i}} V_{x_i} \times W_\alpha = X \times Y.$$

Note that the collection $\{(x_i, \alpha) \text{ such that } \alpha \in F_{x_i}\}$ is a finite set, while $V_{x_i} \subset V_\alpha$. Hence we have found a finite subcover:

$$\{V_\alpha \times W_\alpha\}_{\{(x_i,\alpha) \text{ such that } \alpha \in F_{x_i}\}}$$

(II) Now, for the general case. If $\mathcal{U} = \{U_\beta\}$ is an arbitrary open cover of $X \times Y$, for every β, let us choose a set $\mathcal{C}_\beta$ and open subsets of X and of Y so that

$$U_\beta = \bigcup_{\gamma \in \mathcal{C}_\beta} V_\gamma \times W_\gamma. \tag{20.3.0.2}$$

Let $\mathcal{A} = \bigcup_{\beta \in \mathcal{B}} \mathcal{C}_\beta$; then we have an open cover

$$\{V_\gamma \times W_\gamma\}_{\gamma \in \mathcal{A}}.$$

We produced a finite subcover of such a collection in (I). So let $\mathcal{A}' \subset \mathcal{A}$ be the finite subset for which

$$\{V_\gamma \times W_\gamma\}_{\gamma \in \mathcal{A}'}$$

is an open cover of $X \times Y$. For every $\gamma \in \mathcal{A}'$, there exists some $\beta(\gamma) \in \mathcal{B}$ for which

$$V_\gamma \times W_\gamma \subset U_{\beta(\gamma)}$$

by design (20.3.0.2). Thus we find

$$X \times Y \subset \bigcup_{\gamma \in \mathcal{A}'} U_\gamma \times W_\gamma \subset \bigcup_{\beta(\gamma)} U_{\beta(\gamma)} \subset X \times Y.$$

In other words, the collection

$$\{U_{\beta(\gamma)}\}$$

is a finite subcover of $\mathcal{U}$.

□