Proofs of Propositions and selected exercises (Lectures 16 - 20)

Proposition 16.1.0.2 Let $(X, \mathcal{T})$ be a topological space. For this problem, we let $\mathcal{K}$ denote the collection of closed subsets. Show the following are true:

1. $\emptyset, X \in \mathcal{K}$.

2. If $A_1, \ldots, A_n \in \mathcal{K}$ is a finite collection, then $\bigcup_{i=1,\ldots,n} A_i$ is in $\mathcal{K}$.

3. For an arbitrary collection $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ of elements of $\mathcal{K}$, we have that the intersection $\bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is also in $\mathcal{K}$.

Proof of Proposition 16.1.0.2. Note that in this problem, given $A \subset X$, the complement of A is the complement of A in X. That is, if A^C denotes the complement, we have

$$A^C = \{x \in X \text{ such that } x \notin A.\}.$$

1. To show the empty set is closed, we must show its complement is open. We know $\emptyset^C = X$, and by the definition of topological space, we know X is open. This shows that the empty set is closed.

To show that X is closed, we must show that its complement is open. We know $X^C = \emptyset$, and the empty set is always open (by definition of topological space). Thsi shows that X is closed.

2. Since each A_i is closed, we know A_i^C is open for every $i = 1, \ldots, n$. By DeMorgan's Laws, we have

$$(\bigcup_{i=1,\ldots,n} A_i)^C = \bigcap_{i=1,\ldots,n} (A_i^C).$$

The righthand side is a finite intersection open sets. Hence it is open (by definition of topological space). Because $(\bigcup_{i=1,\ldots,n} A_i)^C$ is open, we conclude that $\bigcup_{i=1,\ldots,n} A_i$ is closed (by definition of closed set). This finishes the proof.

3. Again by DeMorgan's Laws, we have

$$(\bigcap_{\alpha \in \mathcal{A}} A_\alpha)^C = \bigcup_{\alpha \in \mathcal{A}} (A_\alpha^C).$$

Each A_α^C is open because each A_α is closed; thus the righthand side is a union of open sets. Thus the righthand side is open (by definition of topological space). This shows that $(\bigcap_{\alpha \in \mathcal{A}} A_\alpha)^C$ is open, which means $\bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is closed (by definition of closed set). □

Proposition 16.1.0.3 Let $f : X \to Y$ be a continuous map of topological spaces. Show that if $A \subset Y$ is closed, then its preimage is closed.

Conversely, suppose that $f : X \to Y$ is a function such that whenever $A \subset Y$ is closed, its preimage is closed. Prove that f is continuous.

Proof of Proposition 16.1.0.3. Let $A \subset Y$ be closed. Then A^C is open. Moreover,

$$f^{-1}(A^C) = f^{-1}(A)^C.$$

(To see this, you need to exhibit each set as a subset of the other. Well, x is in the lefthand side if $f(x) \notin A$. In particular, $x \notin f^{-1}(A)$. Likewise, if x is in the righthand side, then $x \notin f^{-1}(A)$, so $f(x) \notin A$, meaning $x \in f^{-1}(A^C)$.)

And the lefthand side is open by definition of continuous map. Thus $f^{-1}(A)^C$ is open, meaning $f^{-1}(A)$ is closed. □

Proposition 16.1.0.4. Let $B \subset X$ be an arbitrary subset. Show that there exists a subset, $\overline{B} \subset X$, satisfying the following properties:

1. $B \subset \overline{B}$
2. $\overline{B}$ is closed.
3. Moreover, if C is any other closed subset of X containing B, then C contains $\overline{B}$.

Proof of Proposition 16.1.0.4. Given B, let $\mathcal{K}_B$ denote the collection of all subsets $K \subset X$ for which (i) K is closed, and (ii) K contains B. Note that $\mathcal{K}_B$ is non-empty because it contains X. Now we define

$$\overline{B} := \bigcap_{K \in \mathcal{K}_B} K.$$

We see $\overline{B}$ is closed because arbitrary intersections of closed sets are closed (proving 2.). We also see that $\overline{B}$ contains B because B is contained in every $K \in \mathcal{K}_B$ (proving 1). Finally, if C is any other closed subset of X containing B, then $C \in \mathcal{K}_B$, so in particular, $C \supset \bigcap_{K \in \mathcal{K}_B} K$ (proving 3). □

Proposition 16.2.0.2. Let $\{U_\alpha\}$ be an open cover of X. Note there is a function

$$p : \coprod_{\alpha \in \mathcal{A}} U_\alpha \to X.$$

Prove that the induced map

$$(\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \to X$$

is a homeomorphism. (Here, the equivalence relation $\sim$ is the one for which $x \sim x' \iff p(x) = p(x')$.

Proof of Proposition 16.2.0.2. Let us recall the definition of the disjoint union $\coprod_{\alpha,\mathcal{A}} U_\alpha$—this is the set of all pairs (x, α) where $\alpha \in \mathcal{A}$ and $x \in U_\alpha$. Then the function p is given by

$$p : \coprod_{\alpha \in \mathcal{A}} U_\alpha \to X, \qquad (x, \alpha) \mapsto x.$$

That is, $p(x, \alpha) = x$.

Because $\{U_\alpha\}$ is a cover of X, for every $x \in X$, there is some α such that $x \in U_\alpha$. In particular, for every $x \in X$, there is some (x, α) such that $p(x) = x$. This shows that p is a surjection.

In a previous lecture, we showed that whenever p is a surjection, then the induced function

$$\text{domain of } p/\sim \to \text{ codomain of } p$$

is a bijection if $\sim$ is defined by $x \sim x' \iff p(x) = p(x')$. This is exactly the equivalence relation that we are taking, so we conclude that the induced map

$$f : (\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \to X$$

is a bijection. Note that we have now given this map a name: f.

First let's show f is continuous. In homework you showed that f is continuous if and only if p is. (A map from the quotient is continuous if and only if the its composition with the projection map is.) So let us show p is continuous. This means we must show that if $V \subset X$ is an open subset, then $p^{-1}(V)$ is open. By definition of coproduct topology, $p^{-1}(V)$ is open if and only if

$$p^{-1}(V) \cap U_\alpha$$

is open for every $\alpha \in \mathcal{A}$. So let's prove it. First, we compute:

$$\begin{aligned} p^{-1}(V) \cap U_\alpha &= \{ \ x \in U_\alpha \text{ such that } p(x) \in V \} \\ &= \{ \ x \in U_\alpha \text{ such that } x \in V \\ &= V \cap U_\alpha. \end{aligned}$$

Because $V \subset X$ is open and $U_\alpha \subset X$ is open, their (finite) intersection is open. This shows that $p^{-1}(V) \cap U_\alpha$ is open for every $\alpha \in \mathcal{A}$, and hence that $p^{-1}(V)$ is open. This shows p is continuous. Thus f is continuous.

Now we must show that the inverse map

$$g : X \to (\coprod_\alpha U_\alpha)/ \sim$$

is continuous. For this, it suffices to show that if a subset V in the codomain is open, then $g^{-1}(V) = f(V)$ is open. Well, something in the codomain is open if and only if its preimage under the quotient map is open (by definition of quotient topology). Thus, a subset V of the codomain is open if and only if V is the image of some open subset

$$\tilde{V} \subset \coprod_\alpha U_\alpha.$$

Again by definition of coproduct topology, $\tilde{V}$ then has the property that $\tilde{V} \cup U_\alpha$ is open for every α. But then we have

$$f(V) = p(\tilde{V}) = p(\bigcup_\alpha V \cap U_\alpha) = \bigcup_\alpha p(V \cap U_\alpha).$$

For every α, we know $V \cap U_\alpha$ is open an open subset of X (because it is a finite intersection of open sets), so $p(V \cap U_\alpha) = V \cap U_\alpha$ is an open subset of X. In other words, the rightmost term in the above string of equalities is a union of open sets, and is hence open (by definition of topology). This shows $f(V)$ is open, which completes the proof. $\square$

Proposition 17.1.0.3 Let $\mathcal{A} = X \times \mathbb{R}_{>0}$ be the set of pairs (x, r) where $x \in X$ and r is a positive real number. Let (X, d) be a metric space, and equip it with the induced topology.

(i) The collection

$$\mathcal{U} = \{\text{Ball}(x, r)\}_{(x,r)\in\mathcal{A}}$$

is an open cover of X.

(ii) Now let $\mathcal{B} \subset \mathcal{A}$ denote the set of pairs (x, r) where $x \in x$ and r is a positive *rational* number. (So $\mathcal{B} = X \times \mathbb{Q}$.) Then $\{U_\beta\}_{\beta\in\mathcal{B}}$ is a subcover of $\mathcal{U}$.

Proof of 17.1.0.3. (i) By definition of metric topology (i.e., the topology induced by the metric), every $\text{Ball}(x, r)$ is open. Moreover, for ever $x \in X$, clearly $x \in \text{Ball}(x, r)$ for any $r > 0$, so we conclude $X \subset \bigcup_{(x,r)\in\mathcal{A}} \text{Ball}(x, r)$. On the other hand, the union $\bigcup_{(x,r)\in\mathcal{A}}$ is clearly a subset of X, being a union of subsets of X. This proves that $\mathcal{U}$ is an open cover.

(ii) There is a typo; $\mathcal{B}$ does not equal $X \times \mathbb{Q}$, but it equals $X \times \mathbb{Q}_{>0}$—i.e., X times the set of positive rational numbers. Regardless, for any rational positive number r and any $x \in X$, we have that $x \in \text{Ball}(x, r)$, so we again have that $X = \bigcup_{(x,r)\in\mathcal{B}} \text{Ball}(x, r)$. This proves the claim. □

Exercise 17.0.0.3 Show that this definition is equivalent to the old one: " We say that a collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of subsets of X is a *cover* if $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = X$. We further say this collection is an *open cover* if each U_α is open."

Solution to Exercise 17.0.0.3. The notation $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ means that we have some set $\mathcal{A}$, and for every $\alpha \in \mathcal{A}$, we have specified some open subset $U_\alpha \subset X$. That is the same information as giving a function from $\mathcal{A}$ to $\mathcal{T}$. And of course, if $U_\alpha \in \mathcal{T}$, it is open by definition.

The "cover" part of the definitions are identical, so there is nothing to check there. □

Exercise 17.1.0.2 Let $\mathcal{U}$ be an open cover. Then a subcover of $\mathcal{U}$ is the same data as a choice of subset $\mathcal{B} \subset \mathcal{A}$ such that the composition

$$\mathcal{B} \to \mathcal{A} \to \mathcal{T}$$

is an open cover of X.

Solution to Exercise 17.1.0.2. As stated, the exercise isn't quite correct; we'll see why. Suppose you have an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$.

The first definition (17.1.0.1) says $\{U_\beta\}_{\beta \in \mathcal{B}}$ is a subcover if (i) if the union $\bigcup_{\beta \in \mathcal{B}} U_\beta$ is equal to X, (ii) for every β, there is an α so that $U_\alpha = U_\beta$.

The second definition (17.1.0.2) is identical for (i). Above, (ii) says we can find a function $i : \mathcal{B} \to \mathcal{A}$ so that $U_{i(\beta)} = U_\alpha$.

□

Exercise 18.1.0.1 Show that addition,

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x_1, x_2) \mapsto x_1 + x_2$$

is continuous. (Here, $\mathbb{R}$ is given the topology induced by the standard metric.)

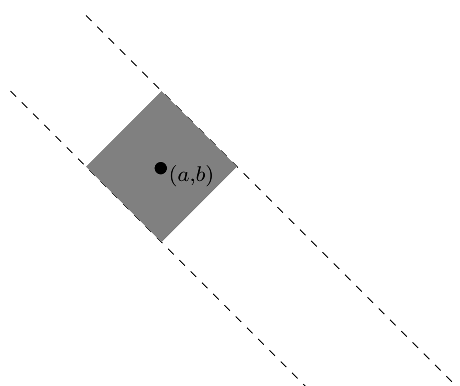
Proof of 18.1.0.1. For notation's sake, let's call the addition function f, so that $f(x_1, x_2) = x_1 + x_2$. We will use the ϵ-δ criterion to prove that f is continuous.

Fix $x = (a, b) \in \mathbb{R}^2$ and fix $\epsilon > 0$. Then

$$U := f^{-1}((a + b - \epsilon, a + b + \epsilon))$$

is the region in $\mathbb{R}^2$ contained (strictly) between the two lines $x_1 + x_2 = a + b - \epsilon$ and $x_1 + x_2 = a + b + \epsilon$. We must now find δ so that the open ball of radius δ around (a, b) is contained in U.

For this let us use some geometry. Clearly, the open diamond/rhombus of total width 2ϵ and total height 2ϵ, centered at (a, b), is contained in U.



In turn, the open ball of radius $\sqrt{\epsilon/2}$ is contained in this open rhombus. Thus setting $\delta = \sqrt{\epsilon/2}$, we are finished. □

Exercise 18.1.0.2 Show that the multiplication function

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x_1, x_2) \mapsto x_1 x_2$$

is continuous. (Here, $\mathbb{R}$ is given the topology induced by the standard metric.)

Proof of 18.1.0.2. Fix a point $(a, b) \in \mathbb{R}^2$. The note that for any $d \in \mathbb{R}$, we have that

$$(a + d)(b + d) = ab + (b + a)d + d^2.$$

And in particular,

$$d_{\mathbb{R}_{std}}(ab, (a + d)(b + d)) = |(b + a)d + d^2| \leq |b + a||d| + |d|^2.$$

Note that given $\epsilon > 0$, the sum $|b + a||d| + |d|^2$ is less than ϵ if each term of the sum is less than $\epsilon/2$—that is, if

$$|b + a||d| < \epsilon/2 \qquad \text{and} \qquad |d|^2 < \epsilon/2.$$

So let δ be any positive real number such that

$$\delta < \min\{\epsilon/2(|b + a|), \sqrt{\epsilon/2}\}.$$

Then we are finished. □

Exercise 18.1.0.3 Show that the following functions are continuous:

1. Fix a real number $a \in \mathbb{R}$. The constant function

$$\mathbb{R} \to \mathbb{R}, \qquad x \mapsto a.$$

2. Fix two continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. The function

$$\mathbb{R} \to \mathbb{R} \times \mathbb{R}, \qquad x \mapsto (f(x), g(x)).$$

Proof of 18.1.0.3. 1. Given any ϵ, any δ will do.

2. You've shown this in your homework for metric spaces. More generally, let W, X, Y be topological spaces, and fix two continuous function $f : W \to X$ and $g : W \to Y$. We will show that $h : W \to X \times Y, h(w) := ((f(w), g(w)))$, is continuous.

Let $A \subset X \times Y$ be open. By definition (of product topology),

$$A = \bigcup_{\alpha \in \mathcal{A}} U_\alpha \times V_\alpha$$

for some set $\mathcal{A}$, and where $U_\alpha \subset X$ and $V_\alpha \subset Y$ are open. Note that

$$h^{-1}(U_\alpha \times V_\alpha) = f^{-1}(U_\alpha) \cap g^{-1}(V_\alpha).$$

Because f and g are continuous, we see that $h^{-1}(U_\alpha \times V_\alpha)$ is thus an intersection of two open sets—thus, $h^{-1}(U_\alpha \times V_\alpha)$ is open. We conclude that

$$h^{-1}(W) = \bigcup_{\alpha \in \mathcal{A}} h^{-1}(U_\alpha \times V_\alpha)$$

so $h^{-1}(W)$ is an open subset of X (being a union of open subsets). This concludes the proof. □

Exercise 18.2.0.1 (You will need to rely on the exercises above. If you want, you can try proving the following propositions *without* proving the exercises yourself, but taking their truth for granted.)

1. Any polynomial function in one variable is continuous. That is, if one has a finite collection of real numbers $a_0, \ldots, a_n$, the function

$$p : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n = \sum_{i=0}^{n} a_i x^i$$

   is continuous. (Hint: Induction on n.)

2. Any polynomial function in finitely many variables is continuous. That is, if we are given a real number $a_{i_1,\ldots,i_m}$ for some finite collection of m-tuples of non-negative integers $i_1, \ldots, i_m$, the function

$$\mathbb{R}^m \to \mathbb{R}, \qquad (x_1, \ldots, x_m) \mapsto \sum_{i_1,\ldots,i_m} a_{i_1,\ldots,i_m} x_1^{i_1} \ldots x_m^{i_m}$$

   is continuous. (Hint: A lot of induction.)

Proof of 18.2.0.1. 1. First, let us prove that the function $f_n : x \mapsto x^n$ is continuous. We will perform induction on the degree n. For $n = 1$ this is obvious. For $n = 2$, we note that $f_n(x) = f_1(x) \cdot f_{n-1}(x)$. This is the composition

$$\mathbb{R} \xrightarrow{(f_1, f_{n-1})} \mathbb{R} \times \mathbb{R} \xrightarrow{\text{multiplication}} \mathbb{R}.$$

The second arrow is continuous by Exercise 18.1.0.2. the first arrow is continuous by Exercise 18.1.0.32 and by induction. Because the composition of continuous functions is continuous, we conclude that f_n is continuous given that f_{n-1} is continuous.

Second, let us now note that the function $x \mapsto ax^n$ (for any constant $a \in \mathbb{R}$) is continuous. This function can again be written as a composition

$$\mathbb{R} \xrightarrow{(a, f_{n-1})} \mathbb{R} \times \mathbb{R} \xrightarrow{\text{multiplication}} \mathbb{R}.$$

which is continuous by combining the inductive proof above with Exercise 18.1.0.31.

Finally, we must prove that the polynomial function p is continuous. We proceed by induction by the degree n of p. For $n = 0$, p is the constant function $x \mapsto a_0$. this is continuous by a previous exercise (18.1.0.3 1). Now suppose that any polynomial q of degree $n - 1$ is continuous. Then p can be written as a composition

$$\mathbb{R} \xrightarrow{(q, a_n x^n)} \mathbb{R} \times \mathbb{R} \xrightarrow{\text{addition}} \mathbb{R}$$

where $q(x) = a_0 + a_1 x^1 + \ldots a_{n-1} x^{n-1}$. Each function in this composition is continuous, hence so is the composition. This completes the proof of 1.

2. Omitted. ☐

Proposition 18.3.0.1

1. Fix a real number $b \in \mathbb{R}$. Then the (singleton) set $\{b\} \subset \mathbb{R}$ is closed.

2. For every $m \geq 1$, the $(m-1)$-dimensional sphere

$$S^{m-1} \subset \mathbb{R}^m$$

is a closed subset of $\mathbb{R}^m$. (Recall that

$$S^{m-1} := \{(x_1, \ldots, x_m) \text{ such that } \sum_{i=1}^{m} x_i^2 = 1\}.$$

As a hint, you can use the fact that for continuous functions, preimages of closed subsets are closed.)

3. More generally, given any polynomial p in m variables, the set

$$\{x \text{ such that } p(x) = 0\} \subset \mathbb{R}^m$$

is a closed subset.

4. Even more generally, given a finite collection of polynomials $p_1, \ldots, p_k$ in m variables, the set

$$\{x \text{ such that } p_i(x) = 0 \text{ for all } i\} \subset \mathbb{R}^m$$

is a closed subset.

5. Even more generally, given an arbitrary collection of polynomials $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ in m variables, the set

$$\{x \text{ such that } p_\alpha(x) = 0 \text{ for every } \alpha \in \mathcal{A}\} \subset \mathbb{R}^m$$

is a closed subset.

Proof of 18.3.0.1. 1. The complement $U = \mathbb{R} \setminus \{b\}$ is open. (For example, for any $x \in U$, the open ball $\text{Ball}(x; |b - x|)$ is contained in U.) This shows that $\{b\} \subset \mathbb{R}$ is closed.

2. Let $p(x_1, \ldots, x_m) = x_1^2 + \ldots x_m^2$. This is a function $p : \mathbb{R}^m \to \mathbb{R}$, and is continuous because it is polynomial. Hence preimages of closed subsets are closed. Now we note that $\{1\} \subset RR$ is closed by the previous part of this problem, and we note that $p^{-1}(\{1\}) = S^{m-1}$.

3. Same proof, but by taking $\{b\} = \{0\} \subset \mathbb{R}$.

4. Given part 3., note that the set in questin is the intersection of $p_i^{-1}(\{0\})$; i.e., an intersection of closed subsets of $\mathbb{R}^m$. Hence it is closed.

4. Same proof. □

Proposition 18.3.0.2

1. Fix a real number a. Then the set

$$(-\infty, a] \subset \mathbb{R}$$

is closed (under the standard topology).

2. Fix a real number a and let $p : \mathbb{R}^m \to \mathbb{R}$ be a polynomial function in m variables. Then the set

$$\{x \in \mathbb{R}^m \text{ such that } p(x) \leq a\ \}$$

is closed. If you need to, do the same for $\geq a$ rather than $\leq a$.

Proof of Proposition 19.1.0.1. 1. The set $U = (a, \infty) \subset \mathbb{R}$ is open. For example, for any $x \in U$, we have that the open ball $\text{Ball}(x; |a-x|)$ is contained in U. This shows $U^C = (-\infty, a]$ is closed.

2. The indicated set is $p^{-1}((-\infty, a])$. Because p is continuous (Exercise 18.2.0.11), and preimages of closed sets are closed sets for continuous maps, the claim follows from the previous part of this problem. □

Proposition 19.1.0.1. Let $d : X \times X \to \mathbb{R}$ be a metric. Endow X with the metric topology (i.e., the topology induced by the metric) and endow $X \times X$ with the product topology. $\mathbb{R}$ has the standard topology.

1. Show that d is continuous.

2. For any $x_0 \in X$, show that the function

$$d(x_0, -) : X \to \mathbb{R}, \qquad x \mapsto d(x_0, x)$$

is continuous.

Proof of 19.1.0.1. 1. We use the ϵ-δ criterion, remembering that the product metric is given by

$$d_{X\times Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

(In this problem, Y happens to equal X.) So fix $(x_1, x_2) \in X \times X$ along with $\epsilon > 0$. For any δ, we have that

$$d_{X\times X}((x_1, x_2), (x_1', x_2')) < \delta \implies d(x_1, x_1') + d(x_2, x_2') < \delta. \qquad (19.3.0.3)$$

Keep the above in mind. Now let's repeatedly apply the triangle inequality:

$$d(x_1', x_2') \leq d(x_1', x_1) + d(x_1, x_2') \leq d(x_1', x_1) + d(x_1, x_2) + d(x_2, x_2'). \qquad (19.3.0.4)$$

By symmetry, we also conclude

$$d(x_1, x_2) \leq d(x_1', x_1) + d(x_1', x_2') + d(x_2, x_2'). \qquad (19.3.0.5)$$

Combining (19.3.0.4) and (19.3.0.5) we obtain:

$$|d(x_1, x_2) - d(x_1', x_2')| \leq d(x_1', x_1) + d(x_2, x_2').$$

By the previous equation (19.3.0.3), we conclude

$$|d(x_1, x_2) - d(x_1', x_2')| \leq 2\delta.$$

Thus choosing δ to be any number less than $\epsilon/2$, we are finished.

2. We note that the function in question is a composition

$$X \to X \times X \xrightarrow{d} \mathbb{R}$$

where the first function sends $x \mapsto (x_0, x)$. So it suffices to prove that for any $x_0 \in X$, the "horizontal inclusion" function

$$X \to X \times, \qquad x \mapsto (x_0, x)$$

is continuous. Because X is a metric space, let us use the ϵ-δ criterion. Given ϵ, let δ be any positive number less than ϵ. Then if $d(x, x') < \epsilon$, we see that

$$d_{X \times X}((x_0, x), (x_0, x')) = d(x_0, x_0) + d(x, x') = 0 + \delta < \epsilon.$$

□

Proposition 19.1.0.2 Let $d : X \times X \to \mathbb{R}$ be a metric. Endow X with the metric topology (i.e., the topology induced by the metric) and endow $X \times X$ with the product topology.

1. Fix a real number $a \in \mathbb{R}$. For every $x_0 \in X$, show that

   $$\{\ x \in X \text{ such that } d(x_0, x) = a\ \}$$

   is a closed subset of X.

2. Fix a real number $a \in \mathbb{R}$. For every $x_0 \in X$, show that

   $$\{\ x \in X \text{ such that } d(x_0, x) \leq a\ \}$$

   is a closed subset of X. This is called the *closed ball of radius a centered at x_0.*

Proof of 19.1.0.2. 1. By Proposition 18.3.0.1, the set $\{a\} \subset \mathbb{R}$ is closed. We know that for all $x_0 \in X$, the function $x \mapsto d(x_0, x)$ is continuous (Proposition 19.1.0.1(2)). Thus the preimage of $\{a\}$ is closed, and the set in question is precisely said preimage.

2. Same exact proof, except we take our closed set in $\mathbb{R}$ to be $(-\infty, a] \subset \mathbb{R}$. (This is closed by Proposition 18.3.0.2(1).) □