Lecture 21

Connectedness

Today is a lecture day. You can drop your pens and pencils.

21.1 From last lecture

21.1.1 On writing proof

(Antoni) Gaudi is the architect who designed the Sagrada Familia in Barcelona, Spain. I paraphrased him in class, saying

> There should be light. But not too much.[1]

(Gaudi was presumably talking about the design of spaces, and how much light a space should have.)

The same applies to proofs. "Light" here is a euphemism for detail. So when I write you a proof, many details may be excluded, so as not to blind you; but the details included should give you sight of what is going on.

But there are two parts to the light advice: Even before worrying about having too much light, you should have light. Many of the proofs I've read in homework lack lighting. So blind me. Overburden me with your light.

[1] The actual quote is "The amount of light should be just right, not too much, not too little, since having too much or too little light can both cause blindness."

21.1.2 Openness and closedness

When we say a subset A is closed or open, it *matters* to specify what space A is a subset of. For example, let us consider the space

$$X = [a, b) \subset \mathbb{R}$$

and endow X with the subspace topology. Let $U = [a, a + \epsilon)$ for some small positive number ϵ (so that in particular, $U \subset X$). Then

U is an open subset of X (you should check this using the definition of the topology of X).

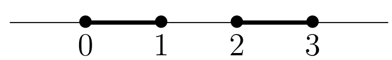
But

U is not an open subset of $\mathbb{R}$ (you should check this also).

21.2 Path-connectedness

We begin with an example.

Example 21.2.0.1. Let $X = [0, 1] \coprod [2, 3] \subset \mathbb{R}$, drawn below:



Would you call X connected?

Remark 21.2.0.2 (Properties of spaces vs. properties of subsets)**.** Above, I used that X was a subset of $\mathbb{R}$ to define the topology of X, but once we know about X's topology, we could ask the connectedness question of X (without reference to $\mathbb{R}$). Is the following space connected?

(Importantly, the picture makes no reference to $\mathbb{R}$ itself.) So unlike "closed" or "open," the adjective "connected" makes sense as a property of a space X. And when we ask whether a subset is connected, we are asking about the property of that subset as a space (endowed with the subspace topology). Aside from specifying the topology of the subspace, the parent set is irrelevant to the question of connectedness.

I want to talk today about two different ways to talk about the *connectedness* of a topological space.

This is the most intuitive definition. First, some preliminaries: We let

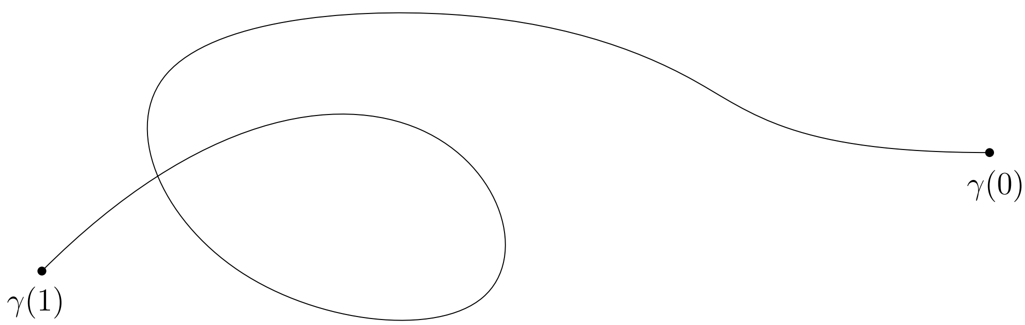
$$[0, 1]$$

denote the usual closed interval from 0 to 1. We treat it as a topological space by giving it the subspace topology inherited from $\mathbb{R}$.

Definition 21.2.0.3. Let X be a topological space. A *path* in X is a continuous function

$$\gamma : [0, 1] \to X.$$

Example 21.2.0.4. Below is an image of a possible path $\gamma : [0, 1] \to \mathbb{R}^2$.



Note that a path need not be injective (it can cross over itself).

Definition 21.2.0.5. Let X be a topological space, and fix a path $\gamma : [0, 1] \to X$. We say that γ is a path *from* $\gamma(0)$ *to* $\gamma(1)$.

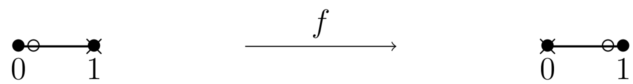
Proposition 21.2.0.6. Let X be a topological space, and fix $x, x' \in X$. If there exists a path from x to x', then there exists a path from x' to x.

This should be an intuitive proposition: If there's a path from x to x', you can just "reverse" the path to get from x' to x. That's the intuition we'll follow in the proof.

Proof. Consider the function

$$f : [0, 1] \to [0, 1], \qquad t \mapsto 1 - t.$$

(So for example, $f(0) = 1$ and $f(1) = 0$.)



You can check that f is continuous.

Now, let
$$\gamma : [0,1] \to X$$
be a path from x to x' (so $\gamma(0) = x$, and $\gamma(1) = x'$). Let us define
$$\overline{\gamma} = \gamma \circ f.$$
Because f and γ are continuous, the composition $\overline{\gamma}$ is. Moreover,
$$\overline{\gamma}(0) = \gamma(f(0)) = \gamma(1) = x'$$
and likewise, $\overline{\gamma}(1) = x$. Thus $\overline{\gamma}$ is a path from x' to x. $\square$

Remark 21.2.0.7. Let X be a topological space. Then for any $x \in X$, there exists a path from x to itself. To see this, note that the constant path
$$\gamma : [0,1] \to X, \qquad \gamma(t) = x \forall t \in [0,1]$$
is a path from x to itself.

The previous proposition says that if there is a path from x to x', then there is a path from x' to x.

Moreover, it turns out you can prove that if there is a path from x to x', and if. there is a path from x' to x'', then there is a path from x to x''. To see this, suppose we have two paths
$$\gamma' : [0,1] \to X, \gamma'' : [0,1] \to X$$
such that $\gamma'(0) = x$, $\gamma'(1) = \gamma''(0) = x'$, and $\gamma''(1) = x''$. Define a path as follows:
$$\gamma : [0,1] \to X, \qquad \gamma(t) = \begin{cases} \gamma'(2t) & t \in [0,1/2] \\ \gamma''(2t-1) & t \in [1,2,1]. \end{cases}$$
It would take us a little bit afield to prove that γ is continuous, but I promise you can prove it with the tools at your disposal. Note that
$$\gamma(0) = \gamma'(2 \cdot 0) = \gamma'(0) = x, \qquad \gamma(1) = \gamma''(2-1) = \gamma''(1) = x''$$

so γ is indeed a path from x to x''.

All this is to say that there is an equivalence relation on any topological space X given as follows: We say $x \sim x'$ if and only if there exists a path from x to x'. Though we may not see this too often in this class, there is a name for the set of equivalence classes for this relation:

$$\pi_0(X) = X/\sim .$$

The left-hand side is read "pie nought of X." It is also called the set of "path-connected components" of X.

Definition 21.2.0.8. Let X be a topological space. We say that X is *path-connected* if for any two points $x, x' \in X$, there exists a path from x to x'.

Example 21.2.0.9. Let $X = \mathbb{R}$. Then X is path-connected. To see this, fix any two points $x, x' \in X$. Then define a function γ by "drawing a straight path from x to x'." The previous sentence was vague, so let's make it precise: Define

$$\gamma : [0,1] \to X, \qquad \gamma(t) = x + t(x' - x).$$

Note that x and x' are constants (we've fixed them!) while t is the variable.

γ is a continuous function. Let's shed some light on why: Because we've given $[0,1]$ the subspace topology, the inclusion

$$[0,1] \to \mathbb{R}, \qquad t \mapsto t$$

is a continuous function. Now let $f : \mathbb{R} \to \mathbb{R}$ be the linear function $t \mapsto x + t(x' - x)$. This is continuous (for example, by previous lectures). Hence the composition

$$[0,1] \to \mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous. On the other hand, this composition is precisely γ.

Finally, note that $\gamma(0) = x$ and $\gamma(1) = x'$.

Example 21.2.0.10. More generally, let $X = \mathbb{R}^n$. Then X is path-connected. To see this, given x and x' in X, again define

$$\gamma : [0,1] \to \mathbb{R}^n, \qquad t \mapsto x + t(x' - x).$$

Note now that we are using vector scaling and vector addition/subtraction to define γ. I'll leave it to you to check that γ is continuous, and that γ is a path from x to x'.

By definition, the notion of path-connectedness depends on the topology of $[0,1]$ (because we need to know which functions out of $[0,1]$ are continuous). So let's see something basic about the topology of $[0,1]$:

Proposition 21.2.0.11. Suppose that $A \subset [0,1]$ is a subset which is both closed and open. Then A is either empty, or equal to $[0,1]$.

For this, we'll use a Lemma:

Lemma 21.2.0.12. If $B \subset [0,1]$ is open, and if $b \in B$ does not equal 0 or 1, then there exists some $\epsilon > 0$ so that $(b-\epsilon, b+\epsilon) \subset B$.

Proof of Lemma 21.2.0.12. Since $B \subset [0,1]$ is open, by definition of subspace topology, there exists $W \subset \mathbb{R}$ open so that $B = W \cap [0,1]$. Now consider the intersection $W \cap (0,1)$. This is an open subset of $\mathbb{R}$, being the intersection of two open subsets—in particular, for any $b \in W \cap (0,1)$, there exists an open ball fully contained in $W \cap (0,1)$ containing b. Let ϵ be the radius of this open ball. Then

$$(b-\epsilon, b+\epsilon) = \mathrm{Ball}(b;\epsilon) \subset W \cap (0,1) \subset W \cap [0,1] = B.$$

□

Proof of Proposition 21.2.0.11. First, let us recall a fact from real analysis:

> If $B \subset \mathbb{R}$ is a closed, non-empty, and bounded subset, then B has a minimal element. That is, there exists $b_0 \in B$ such that $b \in B \implies b_0 \leq b$.[2] Likewise, B has a maximal element.

We proceed by contradiction.

Let $B \subset A$ be a closed and open subset; by way of contradiction, we may assume neither B nor B^C are empty. So let us assume $0 \in B$ without loss of generality. (If $0 \notin B$, just swap the roles of B and B^C.)

Let $b_0 = \min B^C$. (Note that B^C is closed and bounded, so it has a minimum by the above fact.) Note also that $b_0 \neq 0$. Moreover, $b_1 \neq 1$—for if so, then $B^C = \{1\} \subset [0,1]$, and B^C is not an open subset of $[0,1]$.

Thus, we may use Lemma 21.2.0.12 to conclude that B^C must contain some interval $(b_0 - \epsilon, b_0 + \epsilon)$. This contradicts the minimality of $b_0 \in B^C$.

Thus, it must be that either B or B^C are empty. This completes the proof. □

[2] By the completeness axiom of the real line, take b_0 to be the infimum of the bounded set B. Then b_0 is a limit point; but B is closed, so $b_0 \in B$.

This proposition is powerful. For example, we have the following:

Corollary 21.2.0.13. Let X be a discrete topological space and fix elements $x, x' \in X$. Then there exists a path from x to x' if and only if $x = x'$.

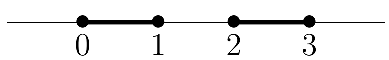
Proof. Suppose $\gamma : [0,1] \to X$ is continuous, and that x is in the image of γ. because X has the discrete topology, the singleton set $\{x\}$ is both closed and open. (To see this, recall that every subset of X is open in the discrete topology. In particular, both $\{x\}$ and its complement are open.) Thus, the preimage $\gamma^{-1}(\{x\})$ is both a closed and open subset of $[0,1]$. By Lemma 21.2.0.11, the preimage must be either empty or all of $[0,1]$. Because we assumed x to be in the image,

$$\gamma^{-1}(\{x\}) = [0,1].$$

In particular, γ is a constant function, so $\gamma(0) = \gamma(1) = x$. $\square$

Example 21.2.0.14. So, if X is a discrete topological space with two or more elements, X is not path-connected.

Example 21.2.0.15. Let $X = [0,1] \coprod [2,3] \subset \mathbb{R}$, drawn below as before:



Then X is not path-connected.

Indeed, I'll take x to be some point in $[0,1]$ and x' to be some point in $[2,3]$. Suppose (for the purpose of contradiction) that there is a path

$$\gamma : [0,1] \to X$$

from x to x'. Then the composition

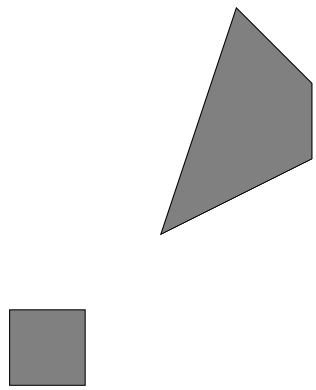
$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}$$

(where the second map is the inclusion map) is continuous. By the intermediate value theorem from calculus, for any value y such that $x \leq y \leq x'$, there must be some $t \in [0,1]$ such that $f(t) = y$.

But γ has image contained in X, and in particular, the composition f has no image in the open interval $(1,2)$. In particular, we have been led to a contradiction.

Remark 21.2.0.16. Note that we have used many results from your analysis class. This is because of the central role of the real line in these discussions, and because your analysis class is devoted to the study of the real line.

Example 21.2.0.17. Let X be the subset of $\mathbb{R}^2$ drawn below, given the subspace topology:

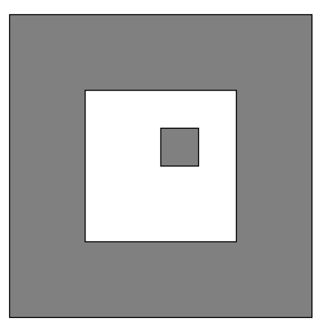


Then X is not path-connected. The proof is similar as the previous example, so I will be brief: By way of contradiction, suppose $\gamma : [0,1] \to X$ is a continuous path from x to x', where x is in the lower-right component of X and x' is in the upper-left component. Then consider the composition

$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}^2 \to \mathbb{R}$$

where the middle arrow is the inclusion, and the last arrow is the projection map sending $(x_1, x_2) \mapsto x_1$. Then f is continuous, being a composition of continuous functions; but again, f will violate the intermediate value theorem.

Example 21.2.0.18. Let X be the subset of $\mathbb{R}^2$ shaded below, given the subspace topology:



Then X is not path-connected. The proof is similar as the previous example, so I will be brief: By way of contradiction, suppose $\gamma : [0,1] \to X$ is a continuous path from x to x', where x is in the middle component of X and x' is in the outer component. Then consider the composition

$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}^2 \to \mathbb{R}$$

where the middle arrow is the inclusion, and the last arrow is now the map sending an element $y \in \mathbb{R}^2$ to the number $d(x,y)$. Then f again violates the intermediate value theorem.

21.3 Connectedness

So, path-connectedness was an intuitive notion: We'll say a space is path-connected if any two points can be connected by a path. Confusingly, the term "path-connected" is not the same as the term "connected" in our culture.

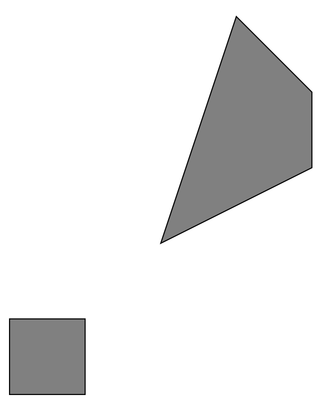
We now discuss a far less intuitive notion:

Definition 21.3.0.1. We say that a space X is *connected* if the following holds: If $A \subset X$ is both open and closed, then either $A = X$ or $A = \emptyset$.

Example 21.3.0.2. By Proposition 21.2.0.11, we know that $X = [0,1]$ is a connected space.

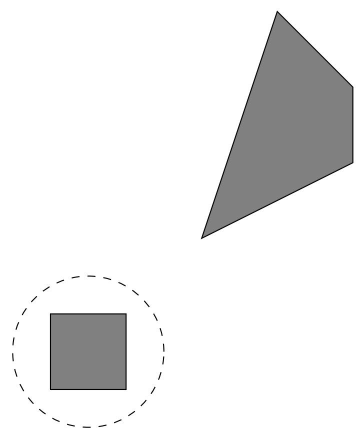
Example 21.3.0.3. Let X be a discrete topological space. If X has two or more elements, X is not connected.

Example 21.3.0.4. Let X be the subset of $\mathbb{R}^2$ drawn below, given the subspace topology:

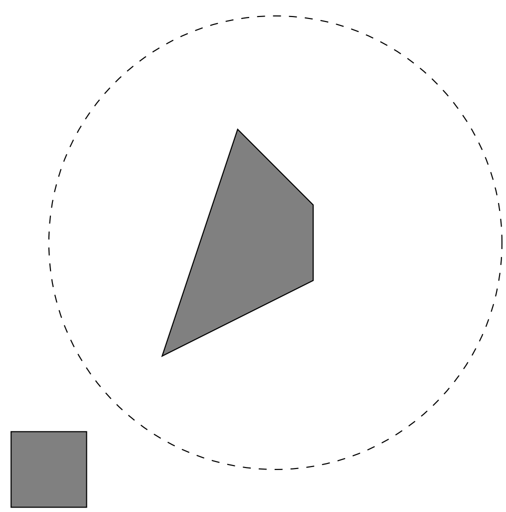


Let us label the lower-left component by A, and the upper-right component by B. I claim that both A and B are each both open and closed.

To see that A is open, simply observe that there is an open ball $W \subset \mathbb{R}^2$ for which $W \cap X = A$ (and then cite the definition of the subspace topology, which defines the topology on $X \subset RR^2$):



Because $B = A^C \subset X$, we conclude B is closed. To see B is open, likewise observe an open ball in X containing B but not A:



So B is open, meaning $A = B^C$ is closed. This shows $A \subset X$ is both open and closed, but $A \neq X$ and $A \neq \emptyset$.

Notice that all our examples connectedness/path-connectedness are the same. This is because of the following:

Proposition 21.3.0.5. If X is path-connected, then X is connected.

Proof. We will prove the contrapositive—that is, if X is not connected, then X is not path-connected.

Because X is not connected, there exists a subset $A \subset X$ which is non-empty, not all of X, but both open and closed.

So choose $x \in A$, and choose $x' \in A^C \subset X$. I claim there is no path from x to x'.

To see this, suppose we have a continuous map $\gamma : [0,1] \to X$ for which γ intersects A, we must have that $\gamma^{-1}(A)$ is non-empty. On the other hand, A is both open and closed, so $\gamma^{-1}(A)$ is both open and closed—this means $\gamma^{-1}(A) = [0,1]$ by Proposition 21.2.0.11.

That is, if $\gamma(t) \in A$ for some t, then $\gamma(t) \in A$ for every $t \in [0,1]$. In particular, if $x = \gamma(0)$, then $x' \neq \gamma(1)$. This proves the claim, and hence the proposition. □

Warning 21.3.0.6. There exist connected spaces that are not path-connected.