Lecture 22

More on connectedness

22.1 Some basics

Let's make explicit the following:

Proposition 22.1.0.1 (Inclusions are continuous.)**.** Let $A \subset X$ and give A the subspace topology. Then the inclusion function $\iota : A \to X$ given by $\iota(a) = a$ is continuous.

Proof. Suppose $W \subset X$ is open. Then $\iota^{-1}(W) = A \cap W$. This is open by definition of subspace topology. □

Proposition 22.1.0.2 (Maps to images of continuous maps are continuous)**.** Let $f : X \to Y$ be continuous, and endow $f(X) \subset Y$ with the subspace topology. Then the function $X \to f(X)$ sending $x \mapsto f(x)$ is continuous.

Proof. Suppose $V \subset f(X)$ is open. Then there exists some subset $W \subset Y$ for which $V = W \cap f(X)$. In particular, $f^{-1}(V) = f^{-1}(W)$. The latter is open in X by definition of continuity, so $f^{-1}(V) \subset X$ is open. □

Proposition 22.1.0.3 (Subspace topologies factor)**.** Let $X \subset Y \subset Z$ and let Z be a topological space. Then the following topologies on X are equal:

- The subspace topology $\mathcal{T}_{X \subset Z}$ of X as a subset of Z.
- The subspace topology $\mathcal{T}_{X \subset Y}$ of X as a subset of Y (where Y is given the subspace topology, induced by virtue of Y being a subset of Z).

Proof. ($\mathcal{T}_{X\subset Z} \subset \mathcal{T}_{X\subset Y}$). Let $U \in \mathcal{T}_{X\subset Z}$. Then by definition, there exists some $W \subset Z$ open for which $U = X \cap W$.

then we see that $U = X \cap W = X \cap (Y \cap W)$, where the last equality is true because $X \subset Y$. By definition of subspace topology (for Y), we see that $V = Y \cap W$ is an open subset of Y. Then $U = X \cap V$ implies that $U \in \mathcal{T}_{X\subset Y}$.

($\mathcal{T}_{X\subset Y} \subset \mathcal{T}_{X\subset Z}$). If $U \in \mathcal{T}_{X\subset Y}$, there is some open subset $V \in \mathcal{T}_Y$ for which $U = V \cap X$. By definition of subspace topology (for Y), we know there exists some $W \subset Z$ open so that $W \cap Y = V$. Hence

$$U = X \cap V = X \cap (W \cap Y) = W \cap (X \cap Y) = W \cap X$$

meaning $U \in \mathcal{T}_{X\subset Z}$.

This finishes the proof. □

22.2 An application of connectedness

Let's recall some ideas from last time. We saw two very different-looking notions of connectedness:

Definition 22.2.0.1 (Path-connected)**.** Let X be a topological space. We say X is path-connected if for every $x, x' \in X$, there exists a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = x'$.

Definition 22.2.0.2 (Connected)**.** Let X be a topological space. We say X is connected if the following hold: If $A \subset X$ is both open and closed, then A is either $\emptyset$ or X.

We also saw:

Proposition 22.2.0.3. The interval $[0,1]$ is connected.

Let's see one application of the idea of connectedness. The intuition for the following is that if a function f is continuous, it does not tear apart things that are connected.

Proposition 22.2.0.4 (Continuous functions preserve connectedness)**.** Let $f : X \to Y$ be a continuous function. If X is connected, then $f(X)$ is connected. If X is path-connected, then so is $f(X)$.

(Note that $f(X) \subset Y$ is being given the subspace topology.)

Proof of Proposition 22.2.0.4. Let $A \subset f(X)$ be both open and closed. Then $f^{-1}(A) \subset X$ is both open and closed. (This is because the map $X \to f(X)$ is closed by the Lemma.) Hence $f^{-1}(A)$ must either be X or $\emptyset$. The former means that A must equal $f(X)$. The latter means that A must be empty by definition of the image $f(X)$.

Hence $f(X)$ is connected.

As for path-connectedness: Let $y, y' \in f(X)$. Choose $x, x' \in X$ so that $f(x) = y$ and $f(x') = y'$. Because X is path-connected, there exists $\gamma : [0,1] \to X$ for which $\gamma(0) = x$ and $\gamma(1) = x'$. Now let $\gamma' = f \circ \gamma$. This is continuous because f and γ are. Moreover,

$$\gamma'(0) = f(\gamma(0)) = y, \qquad \gamma'(1) = f(\gamma(1)) = y'.$$

□

Exercise 22.2.0.5. Prove the following: If X is connected and Y is not, there exists no continuous surjection from X to Y.

Likewise, if X is path-connected but Y is not, there exists no continuous surjection from X to Y.

Exercise 22.2.0.6. Show that if X is connected, then for any equivalence relation $X/\sim$, the quotient space $X/\sim$ is connected.

Likewise, show that if X is path-connected, then for any equivalence relation $X/\sim$, the quotient $X/\sim$ is path-connected.

Exercise 22.2.0.7. Show that $\mathbb{R}P^2$ is path-connected and connected.

22.3 Connectedness is not path-connectedness

Last time we saw that if X is path-connected, then it is connected. We will see that the converse does not hold.

Definition 22.3.0.1 (Topologist's sine curve)**.** We let X be the following union:

$$\{(x_1, x_2) \mid x_1 = 0\} \bigcup \{(x_1, x_2) \mid x_1 > 0 \text{ and } x_2 = \sin(1/x_1)\ \} \subset \mathbb{R}^2.$$

We endow X with the subspace topology (inherited from $\mathbb{R}^2$). We call X the *topologist's sine curve.*

Exercise 22.3.0.2. Draw X (in $\mathbb{R}^2$).

Remark 22.3.0.3. The name "topologist's sine curve" is popular, but probably insinuates an immature separation of mathematical subjects. This space is no more a topologist's than anybody else's.

Theorem 22.3.0.4. X is connected, but it is not path-connected.

Before we prove the theorems, let's set some notation. We set

$$A = \{(x_1, x_2) \mid x_1 = 0\}$$

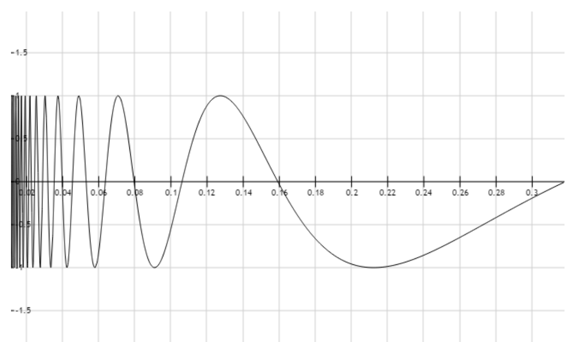
and

$$B = \{(x_1, x_2) \mid x_1 > 0 \text{ and } x_2 = \sin(1/x_1)\ \}$$

so we have that

$$X = A \cup B.$$

Remark 22.3.0.5. The following is what B looks like:



What happens as the x_1 coordinate approaches zero? One description might be that the x_2 coordinate oscillates—and it oscillates "faster and faster".

Lemma 22.3.0.6. A and B are both homeomorphic to $\mathbb{R}$.

Proof. I claim that the function

$$\mathbb{R} \to A, \qquad x \mapsto (0, x)$$

is a homeomorphism. I will leave the proof to you.

As for B, consider the maps

$$\mathbb{R}_{>0} \to B, \qquad t \mapsto (t, \sin(1/t))$$

and

$$B \to \mathbb{R}_{?0}, \qquad (x_1, x_2) \mapsto x_1.$$

These are both continuous and are inverses to each other. Moreover, $\mathbb{R}_{>0}$ is homeomorphic to $\mathbb{R}$ by taking, for example, the log and exp maps. This concludes the proof. □

Lemma 22.3.0.7. if U is an open subset of X and contains all of A, then it intersects B.

Proof. Let $U \subset X$ be open. If $A \subset U$, let $x = (0, 1) \in A$. Because U is an open subset of X, by definition of subspace topology, U is the intersection $W \cap X$ for some open subset $W \subset \mathbb{R}^2$. In particular, there must be some $\delta > 0$ so that the open ball of radius δ centered at x is contained in W. But given any δ¡ there exists some number positive $x_1' < \delta$ so that $1/x_1'$ is an integer multiple of $\pi/2$; in particular, there is some positive $x_1' < \delta$ so that

$$(x_1', \sin(1/x_1')) \in \text{Ball}(x, \delta).$$

That is, $\text{Ball}(x, \delta) \cap B$ is non-empty. This shows that for any $W \subset \mathbb{R}^2$ for which $W \cap X \supset A$, we have that $W \cap B \neq \emptyset$. That is, W intersects B, so U intersects B as well. □

Lemma 22.3.0.8. If K is a closed subset of X that contains all of B, then it must intersect A.

We will see a proof of this next time, when we discuss closures and closed subsetes of metric spaces.

Corollary 22.3.0.9. X is connected.

Proof. Let $Q \subset X$ be open and closed. Let us suppose Q is not empty. We are finished if we can show $Q = X$.

Because Q is non-empty, it contains some element x.

Let us suppose $x \in A$. Then $Q \cap A$ is non-empty; but because Q is both open and closed (in X), we conclude that $Q \cap A$ is both open in closed (in A). Because $A \cong \mathbb{R}$, A is connected; because $Q \cap A$ is non-empty, we conclude that $Q \cap A = A$. In other words, Q contains A. By Lemma 22.3.0.7, $Q \cap B$ is hence non-empty. But then $Q \cap B$ is a non-empty subset of B which is both open and closed; because B is connected (being homeomorphic to $\mathbb{R}$), we conclude that $Q \cap B = B$. But

$$X = A \cup B$$

so we conclude (using $Q \cap A = A$ and $Q \cap B = B$ with $Q \subset X$) that $X = Q$.

On the other hand, if $x \in B$, we again see that $Q \cap B = B$ by connectedness of B. By Lemma 22.3.0.8 we conclude $Q \cap A \neq \emptyset$, and thus $Q \cap A = A$ by connectedness of A. Hence $X = Q$. □

Now, to prove the theorem, it remains for us to prove that X is not path-connected. To that end, let us prove the following:

Lemma 22.3.0.10. Let $[t_0, t_1]$ be a closed interval. Then there does not exist a continuous function

$$f : [t_0, t_1] \to \mathbb{R}^2$$

such that $f(t_0) \in A$ and $f((t_0, t_1]) \subset B$.

Proof. We'll give a proof by contradiction, utilizing the "convergent sequence" criterion for continuity. (A continuous function preserves convergent sequences.)

Suppose a continuous f exists. Consider the composition

$$[t_0, t_1] \xrightarrow{f} \mathbb{R}^2 \xrightarrow{\pi_1 : (x_1, x_2) \mapsto x_1} \mathbb{R}$$

where the last arrow projects to the first coordinate. This composition is continuous (because both the above arrows are continuous).

Let us choose two sequences of real numbers. First, we choose a decreasing sequence

$$s_1, s_2, \ldots, \qquad s_i \in [t_0, t_1]$$

such that $\lim s_i \to t_0$, and so that $\sin(1/f(s_i))$ is constant.

(This can be constructed as follows: One first chooses an arbitrary $s_1 \in (t_0, t_1]$ (in particular, $s_1 \neq t_0$). By assumption, $\pi_1(f(s_1)) > 0$. If we have chosen s_i, by the continuity of f, we can find some

$$s_{i+1}$$

so that

1. $s_{i+1} \in (t_0, (s_i - t_0)/2]$, and
2. $\sin(1/f(s_{i+1})) = \sin(1/f(s_i))$.

By the first condition, the sequence s_i is decreasing and converges to t_0. The second condition ensures that the value $\sin(1/f(s_i)$ is constant with respect to i.)

We choose our second sequence

$$s_1', s_2', \ldots, \qquad s_i' \in [t_0, t_1]$$

again so the sequence is decreasing, so that $\lim s_i' \to t_0$, and so that $\sin(1/f(s_i'))$ is constant, but with the requirement that

$$\sin(1/f(s_i)) \neq \sin(1/f(s_i')).$$

(This inequality can be achieved simply by a prudent choice of s_1'; we are using here that $\pi_2 \circ f$ is non-constant.)

But the composition

$$[t_0, t_1] \xrightarrow{f} \mathbb{R}^2 \xrightarrow{\pi_2 : (x_1, x_2) \mapsto x_2} \mathbb{R}$$

(where we now project to the second coordinate) is also continuous. Thus, we must have that

$$\sin(1/s_1) = \lim \pi_2 \circ f(s_i) = \pi_2 \circ f \lim s_i = \pi_2 \circ f(t_0)$$

(where the middle equality uses the continuity of $\pi_2 \circ f$) and, at the same time,

$$\sin(1/s_1') = \lim \pi_2 \circ f(s_i') = \pi_2 \circ f \lim s_i' = \pi_2 \circ f(t_0).$$

We arrive at a contradiction because $\sin(1/s_1) \neq \sin(1/s_1')$. □

Lemma 22.3.0.11. Let $a \in A$ and $b \in B$. There is no continuous path in X from a to b.

Proof. Let $\gamma : [0,1] \to X$ be continuous. Then $\gamma^{-1}(A) \subset [0,1]$ is a closed subset. (This is because $A \subset X$ is closed—to see this, note that $A \subset \mathbb{R}^2$ is closed.) On the other hand, $\gamma^{-1}(A) \neq [0,1]$ because $\gamma(1) = b \notin A$.

So let $t_0 = \max \gamma^{-1}(A)$ be the largest real number $t \in [0,1]$ for which $\gamma(t) \in A$. Then the composition

$$f : [t_0, 1] \to [0,1] \xrightarrow{\gamma} B \cup \{\gamma(t)\}$$

would be a continuous function contradicting the conclusion of Lemma 22.3.0.10.

□

Now we can finally prove the theorem:

Proof of Theorem 22.3.0.4. We know that X is not path-connected by Lemma 22.3.0.11. So it suffices to show that X is connected. This is the content of Corollary 22.3.0.9. □

22.4 Lessons learned

This lecture contained a lot of new mathematics. The reason we went in-depth was the following: I wanted to show you that a space can be connected, but not path-connected. The proofs above show that the "topologist's sine curve" is exactly such a space.

But there are other results we can glean from above.

Proposition 22.4.0.1. There exist topological spaces Y and continuous functions

$$f : (0, 1] \to Y$$

such that f does *not* extend to a continuous function on $[0, 1]$. that is, one can choose f and Y so that there does not exist a function

$$\gamma : [0, 1] \to Y$$

for which $\gamma(t) = f(t)$ for all $t \in (0, 1]$.

Indeed, even if we demand that $Y = \mathbb{R}^2$ and that f has bounded image, it is not always true that f extends to $[0, 1]$.

Proof. Let $Y = \mathbb{R}^2$. Take f to be the function

$$f(t) = (t, \sin(1/t)).$$

We saw that f does not extend continuously to a function $f : [0, 1] \to \mathbb{R}^2$.

As for the second part of the proposition, notice that the image of f is indeed bounded—for example, the image is contained in the rectangle $[0, 1] \times [-1, 1] \subset \mathbb{R}^2$, which is in turn contained in a ball of radius 3 centered at the origin. □

Remark 22.4.0.2. Another lesson learned is that the difference between "connected" and "path-connected" isn't too pathological.

(“Pathological” is a term that mathematicians use to pass judgement on particular examples. A less judgmental, but equivalent, way to describe a “pathological example:” A pathological examples is one that betrays your early intuitions, and moreover, one having properties that we either rarely encounter, or want to avoid to make proving results easier.)

Note that the topologist’s sine curve is Hausdroff—in fact, it’s even a metric space (being a subspace of $\mathbb{R}^2$). These are the kinds of spaces that we thought we would feel somewhat comfortable with.

It depends on your tastes whether you want to interpret this example as saying ‘’Subspaces of $\mathbb{R}^2$ can be kind of crazy,” or as saying “We should get used to certain phenomena because they will show up whether we expect them or not.”