Lecture 23

Closures and interiors

Fix X a topological space. As you know, given a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of open subsets of X, the union

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$

is an open subset of X. Likewise, given a collection $\{K_\beta\}_{\beta \in \mathcal{B}}$ of closed subsets of X, the intersection

$$\bigcap_{\beta \in \mathcal{B}} K_\beta$$

is a closed subset of X.

Because of these properties, if we fix some subset $B \subset X$, it makes sense to speak of the "large open subset of X contained in B" and "the smallest closed subset of X containing B." These descriptions are informal; we'll make them precise shortly. They characterize the *interior* and *closure* of B, respectively.

These constructions (interiors and closures) are useful, and they're also fun and interesting. Fix some subset $B \subset \mathbb{R}^2$. It's not a bad use of one's day to figure out what the interior and closure of B are.

23.1 Closed subsets of metric spaces

Before we go on, let me prove the following:

Proposition 23.1.1. Let X be a metric space and fix a subset $A \subset X$. Then the following are equivalent

1. A is closed.

2. For every convergent sequence $x_1, \ldots$ such that $x_i \in A$ for every i, then the limit of the sequence is also in A.

Proof. You are proving (1) $\implies$ (2) in your homework. So here we'll prove the converse.

We'll prove (2) $\implies$ (1) by proving the contrapositive. So suppose that A is not closed. Then A^C is not open; so fix $y \notin A$ such that for every $r > 0$, $\text{Ball}(y, r) \not\subset A^C$. (At least one such y is guaranteed to exist if A^C is not open.)

Now fix a decreasing sequence of positive real numbers $r_1, r_2, \ldots$ converging to 0.[1] For every r_i, there exists some $x_i \in A \cap \text{Ball}(y, r_i)$. By construction, $x_1, x_2, \ldots$ is a sequence in A whose limit is y. This proves the contrapositive. □

23.2 Closure

Definition 23.2.1. Fix a topological space X and let $B \subset X$ be a subset.[2] Let

$$\mathcal{K}$$

be the collection of all closed subsets of X containing B.[3] Then the *closure* of B is defined to be

$$\overline{B} := \bigcap_{K \in \mathcal{K}} K.$$

In words, the closure of B is the set obtained by intersecting every closed subset containing B.

Remark 23.2.2. Note that B is always a subset of $\overline{B}$.

Remark 23.2.3. Note that $\overline{B}$ is a closed subset of X. This is because the intersection of closed subsets is always closed.

[1] For example, $r_i = 1/i$.

[2] It could be any kind of subset: open, closed, neither!

[3] Note that X is an element of $\mathcal{K}$.

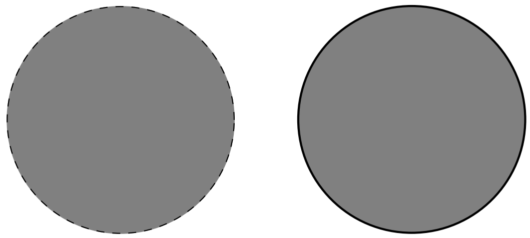

Figure 23.1: An open ball on the right; its closure (a closed ball) on the left.

Remark 23.2.4. If $B \subset X$ is closed, then $\overline{B} = B$. To see this, note that B is an element of $\mathcal{K}$ because B is closed. Hence

$$\bigcap_{K \in \mathcal{K}} K = B \cap \left(\bigcap_{K \in \mathcal{K}, K \neq B} K \right).$$

But this righthand side is a subset of B because it is obtained by intersecting B with some other set. In particular,

$$\overline{B} \subset B.$$

Because $B \subset \overline{B}$ (for any kind of B), we conclude that $B = \overline{B}$.

Example 23.2.5. If $B = \emptyset$, then $\overline{B} = \emptyset$. If $B = X$, then $\overline{B} = X$.

Exercise 23.2.6. Let $X = \mathbb{R}^n$ (with the standard topology). Let $B = \text{Ball}(0, r)$ be the open ball of radius r. Show that the closure of B is the closed ball of radius r; that is,

$$\overline{B} = \{x \in \mathbb{R}^n \text{ such that } d(x, 0) \leq r\ .\}$$

Proof. You are showing in your homework that if $K \subset X$ is closed and if $x_1, \ldots$ is a sequence in K converging to some $x \in X$, then x is in fact an element of K.

Choose a point x of distance r from the origin. And choose also an increasing sequence of positive real numbers $t_1, t_2, \ldots$ converging to 1.[4] Then the sequence

$$x_i = t_i x$$

[4]For example, you could take $t_i = i/(i + 1)$.

is a sequence in B converging to x. If $K \supset B$, then the x_i define a sequence in K; moreover, if K is closed, the limit x is in K. Thus $x \in K$ for any closed subset containing B. In particular, x is in the intersection of all such K. Thus $x \in \overline{B}$. This shows that the closed ball of radius r is contained in $\overline{B}$.

On the other hand, consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $d(0, -)$; that is, the "distance to the origin" function. We see that $f^{-1}([0, r])$ is equal to the closed ball of radius r—in particular, this closed ball is a closed subset of $\mathbb{R}^n$, and it obviously contains $\text{Ball}(0, r)$. This shows that $\overline{B}$ is a subset of the closed ball of radius r (because $\overline{B}$ can be expressed as the intersection of this closed ball with other sets). We are finished. □

Exercise 23.2.7. Suppose $f : X \to Y$ is a continuous function, and let $B \subset X$ be a subset. Show that

$$f(\overline{B}) \subset \overline{f(B)}.$$

In English: The image of the closure of B is contained in the closure of the image of B.

Proof. Let $\mathcal{C}$ be the collection of closed subsets of Y containing $f(B)$. Then

$$f^{-1}(\overline{f(B)}) = f^{-1}\left(\bigcap_{C \in \mathcal{C}} C\right)$$

by definition of closure. We further have:

$$f^{-1}\left(\bigcap_{C \in \mathcal{C}} C\right) = \bigcap_{C \in \mathcal{C}} f^{-1}(C).$$

Now, because f is continuous, we know that $f^{-1}(C)$ is closed for every $C \in \mathcal{C}$. Moreover, because $f(B) \subset C$, we see that $B \subset f^{-1}(C)$. We conclude that for every $C \in \mathcal{C}$, $f^{-1}(C) \in \mathcal{K}$. Thus

$$\bigcap_{K \in \mathcal{K}} K \subset \bigcap_{C \in \mathcal{C}} f^{-1}(C).$$

The lefthand side is the definition of $\overline{B}$. The righthand side is $f^{-1}(\overline{f(B)})$. We are finished. □

Remark 23.2.8. It is not always true that $f(\overline{B})$ is equal to $\overline{f(B)}$. For example, let $B = X = \text{Ball}(0, r)$, and let $f : X \to \mathbb{R}^2$ be the inclusion. Then $f(\overline{B}) = X$, while $\overline{f(B)}$ is the closed ball of radius r.

Exercise 23.2.9. Find an example of a continuous function $p : \mathbb{R}^n \to \mathbb{R}$ such that

$$\overline{\{x \text{ such that } p(x) < t\}},$$

does *not* equal

$$\{x \text{ such that } p(x) \leq t\}.$$

Example 23.2.10. Let $B \subset \mathbb{R}^2$ be the following subset:

$$B = \{(x_1, x_2) \text{ such that } x_1 > 0 \text{ and } x_2 = \sin(1/x_1)\} \subset \mathbb{R}^2.$$

This is not a closed subset of $\mathbb{R}^2$. I claim

$$\overline{B} = B \bigcup \{(x_1, x_2) \text{ such that } x_1 = 0 \text{ and } x_2 \in [-1, 1]\}.$$

That is, $\overline{B}$ is equal to the topologist's sine curve from last class.

Let us call the righthand side S for the time being. First, I claim that $S \subset \overline{B}$. Indeed, fix some point $(0, T) \in S \setminus B$. Then there is an unbounded, increasing sequence of real numbers $t_1, t_2, \ldots$ for which $\sin(t_i) = T$; let $s_i = 1/t_i$. Then the sequence of points

$$x_i = (s_i, \sin(1/s_i)) = (s_i, T)$$

converges to $(0, T)$, while each x_i is an element of B. In particular, $(0, T)$ is contained in any closed subset containing B. This shows $S \subset \overline{B}$.

To complete the proof, it suffices to show that S is closed. For this, because $\mathbb{R}^2$ is a metric space, it suffices to show that any convergent sequence contained in S has a limit contained in S. So let $x_1, x_2, \ldots$ be a sequence in S.

Suppose that the limit $x \in \mathbb{R}^2$ has the property that the 1st coordinate is non-zero. There is a unique point in S with a given non-zero first coordinate t, namely $(t, \sin(1/t))$. Moreover, because the function $t \mapsto \sin(t/1)$ is continuous, if $t_i = \pi_1(x_i)$ converges to t, we know that $(t_i, \sin(1/t_i))$ converges to $(t, \sin(1/t))$. So the limit is in S.

If on the other hand the first coordinate of x is equal to zero, let us examine the second coordinates $\pi_2(x_1), \ldots$. By continuity of π_2, the sequence

$\pi_2(x_1), \pi_2(x_2), \ldots$ converges to some T; because each x_i has a second coordinate in $[-1,1]$, and because $[-1,1] \subset RR$ is closed, we conclude that the limit T is also contained in $[-1,1]$. Hence the limit of the sequence $x_1, \ldots,$ is the point $(0,T)$, and $(0,T) \in S$.

Because any sequence in S with a limit in $\mathbb{R}^2$ has limit in S, S is closed.

23.3 Interiors

Definition 23.3.1. Let X be a topological space and fix $B \subset X$. Let $\mathfrak{U}$ denote the collection of pen subsets of X that are contained in B. Then the *interior* of B is defined to be the union

$$int(B) = \bigcup_{U \in \mathfrak{U}} U.$$

Remark 23.3.2. For any B, we have that $int(B) \subset B$. Moreover, $int(B)$ is an open subset of both B and of X.

Remark 23.3.3. If B is open, then $int(B) = B$. This is because $B \in \mathfrak{U}$, so

$$int(B) = \bigcup_{U \in \mathfrak{U}} U = B \cup \left(\bigcup_{U \neq B, U \in \mathfrak{U}} U \right)$$

meaning $int(B)$ contains B (because $int(B)$ is a union of B with possibly other sets). Thus we have that $int(B) \subset B \subset int(B)$, meaning $int(B) = B$.

Example 23.3.4. We have that $int(\emptyset) = \emptyset$ and $int(X) = X$.

Example 23.3.5. Let $X = \mathbb{R}^n$ and let B be the closed ball of radius r. Then $int(B) = \text{Ball}(0,r)$ is the open ball of radius r.

To see this, we note that $\text{Ball}(0,r)$ is open and contained in B, so $\text{Ball}(0,r) \subset int(B)$ by definition of interior. Because $int(B) \subset B$, it suffices to show that no other point of B (i.e., no point in $B \setminus \text{Ball}(0,r)$) is contained in the interior of B.

So fix $y \in B \setminus \text{Ball}(0,r)$, meaning y is a point of exactly distance r away from the origin. It suffices to show that there is no open ball containing y and contained in B; for then there is no $U \in \mathfrak{U}$ for which $y \in U$.

Well, for any $\delta > 0$, $\text{Ball}(y,\delta) \subset \mathbb{R}^2$ contains some point of distance $> r$ from the origin. So $\text{Ball}(y,\delta)$ is never contained in B. This completes the proof.