Lecture 24

One-point compactification

Today you have a guest instructor. Get into your groups (you know the drill) and tackle the proofs of the propositions below.

Do not spent more than 15 minutes on proving a given proposition. Move on!

Definition 24.0.1. Let X be a topological space. We are now going to create a new topological space X^+.

As a set, $X^+ = X \coprod \{*\}$. In other words, X^+ is the set obtained by adjoining a single point called $*$ to X.

The topology $\mathcal{T}_{X^+}$ is defined as follows: $U \subset X^+$ is open if either

1. $* \notin U$ and U is open in X, or
2. $* \in U$ and $U \cap X$ is the complement of a closed, compact subspace of X.

We call X^+ the *one-point compactification* of X.

Remark 24.0.2. Note that if X is Hausdorff, we may remove the adjective "closed" from the second condition above.

24.1 Basic properties

Prove the following:

Proposition 24.1.1. $\mathcal{T}_{X^+}$ is a topology on the set X^+.

(Thus, you need to prove that the collection of sets U satisfying 1. or 2. satisfies all the properties of a topology. You will want to use at some point that the empty set is a compact space.)

Proposition 24.1.2. X^+ is compact.

(Thus, you need to prove that every open cover of X^+ admits a finite subcover.)

Remark 24.1.3. This justifies the word "compactification."

24.2 Examples

Prove the following:

Proposition 24.2.1. If X is compact, then X^+ is homeomorphic to the coproduct $X \coprod \{*\}$ with the coproduct topology.

Proposition 24.2.2. If $X = \mathbb{R}^n$, then X^+ is homeomorphic to S^n.

Remark 24.2.3. The proof of Proposition 24.2.2 is made easier if you know about *stereographic projection.* This is the function

$$p : S^n \setminus \{(0, \dots, 0, 1)\} \to \mathbb{R}^n, \qquad (x_1, \dots, x_{n+1}) \mapsto \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n).$$

Here is a description of p in words. For brevity, let us call the point $(0, \dots, 0, 1) \in S^n$ the *north pole* of S^n. Given a point $x \in S^n$ such that x is not the north pole, p sends x to the intersection of

- the line through x and the north pole, with
- the hyperplane $\{x_{n+1} = 0\}$, which one can identify with $\mathbb{R}^n$.

Proposition 24.2.4. If X and Y are homeomorphic, so are X^+ and Y^+.