Lecture 25

Density, Interiors

Today you have a guest instructor. Get into your groups (you know the drill) and tackle the proofs of the propositions below.

Do not spent more than 15 minutes on proving a given proposition. Move on!

25.1 Density

Definition 25.1.1. Let X be a topological space and fix a subset $B \subset X$. We say that B is *dense* in X if $\overline{B} = X$.

Prove the following:

Proposition 25.1.2. Fix $B \subset X$. The following are equivalent:

1. G is dense in X.
2. For every open $U \subset X$, $U \cap B \neq \emptyset$.
3. For every $x \in X$, there is some neighborhood A of x in X such that $A \cap B \neq \emptyset$.
4. For every $x \in X$, there is some open neighborhood U of x in X such that $U \cap B \neq \emptyset$.

Proposition 25.1.3. $\mathbb{Q} \subset \mathbb{R}$ is dense.

Proposition 25.1.4. $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 25.1.5. For each of the following examples of subsets of $\mathbb{R}^2$, identify the closure, the interior, and the boundary. Which of these is dense?

1. $B = \{(x_1, x_2) \text{ such that } x_1 \neq 0\ \}$.
2. $B = \bigcup_{(a,b)\in\mathbb{Z}\times\mathbb{Z}}(a-1, a+1) \times (b-1, b+1)$.
3. $B = \{(x_1, x_2)$ such that at least one of the coordinates is rational$\}$.

25.2 Interiors

Definition 25.2.1. Let X be a topological space and fix $B \subset X$. Let $\mathfrak{U}$ denote the collection of pen subsets of X that are contained in B. Then the *interior* of B is defined to be the union

$$int(B) = \bigcup_{U\in\mathfrak{U}} U.$$

Prove the following:

Proposition 25.2.2. For any B, we have that $int(B) \subset B$. Moreover, $int(B)$ is an open subset of both B and of X.

Proposition 25.2.3. $B \subset X$ is open if and only if $int(B) = B$.

Example 25.2.4. We have that $int(\emptyset) = \emptyset$ and $int(X) = X$.

Example 25.2.5. Let $X = \mathbb{R}^n$ and let B be the closed ball of radius r. Then $int(B)$ is the open ball of radius r.

To see this, we note that $\text{Ball}(0, r)$ is open and contained in B, so $\text{Ball}(0, r) \subset int(B)$ by definition of interior. Because $int(B) \subset B$, it suffices to show that no other point of B (i.e., no point in $B \setminus \text{Ball}(0, r)$) is contained in the interior of B.

So fix $y \in B \setminus \text{Ball}(0, r)$, meaning y is a point of exactly distance r away from the origin. It suffices to show that there is no open ball containing y and contained in B; for then there is no $U \in \mathfrak{U}$ for which $y \in U$.

Well, for any $\delta > 0$, $\text{Ball}(y, \delta) \subset \mathbb{R}^2$ contains some point of distance $> r$ from the origin. So $\text{Ball}(y, \delta)$ is never contained in B. This completes the proof.