Lecture 26

Some fun examples

Do not spent more than 10 or 15 minutes on proving a given proposition. Move on and have fun!

26.1 Euclidean space is an open ball

Here is a basic one:

Proposition 26.1.1. Let $r > 0$ and fix $x \in \mathbb{R}^n$. Then $\text{Ball}(x, r)$ is homeomorphic to $\mathbb{R}^n$.

26.2 Tori

Here are four spaces:

1. A is the product space $S^1 \times S^1$.

2. B is the quotient space $[0,1] \times [0,1]/\sim$, where $\sim$ is the following equivalence relation:

$$x \sim x' \iff \begin{cases} x_1, x_1' \in \{0,1\} \text{ and } x_2 = x_2' & \text{or} \\ x_1 = x_1' \text{ and } x_2, x_2' \in \{0,1\} \end{cases}.$$

   Here, $x = (x_1, x_2) \in [0,1] \times [0,1]$. (It may help to draw a picture on a square showing what kind of "gluing" is happening.)

3. C is the surface in $\mathbb{R}^3$ parametrized by the equation

$$\vec{r}(\theta, \phi) = ((a\cos\theta + b)\cos\phi, (a\cos\theta + b)\sin\phi, a\sin\theta)$$

   for $0 < a < b$ and $\theta, \phi \in [0, 2\pi]$.

4. D is the subset in $\mathbb{C} \times \mathbb{C}$ given by points $(x, y) \in \mathbb{C} \times \mathbb{C}$ such that $|x| = |y| = 1$ (given the subspace topology).

Proposition 26.2.1. A and D are homeomorphic.

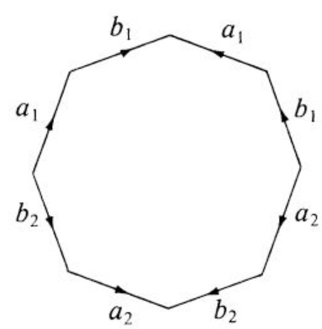
Proposition 26.2.2. B and C are homeomorphic.

Proposition 26.2.3. Any two of the spaces in $\{A, B, C, D\}$ are homeomorphic.

Proposition 26.2.4. If X is any of the above spaces, then for every $x \in X$, there is some open subset $U \subset X$ with $x \in U$ such that U is homeomorphic to $\mathbb{R}^2$.

26.3 What the...?

Can you draw what happens when you glue together the edges of an octahedron as below? (You glue a_1 to the other edge labeled a_1, in the way respecting directions as indicated. Likewise glue edge a_2 to a_2, and b_1 to b_1, and b_2 to b_2.)



Proposition 26.3.1. Let X be the space obtained by gluing as above. For every $x \in X$, there is some open subset $U \subset X$ with $x \in U$ such that U is homeomorphic to $\mathbb{R}^2$.