Notes for Math 4330, General Topology (Fall 2019)

Hiro Lee Tanaka

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Introduction

Welcome to class. Here are some details:

1. Me: Hiro
2. You: Taking Math 4330, General topology
3. My e-mail, office hours, office, handing out the syllabus, et cetera.

Now let's get to the good stuff.

Topology is the study of shapes.

Question 0.0.1. At this point, what questions do you have as a student?

(Here, I field questions. But I move forward with two:)

1. What do you mean by shape?
2. How do you study them?

The purpose of this class is to give you the vocabulary to begin understanding the answers to these questions.

Remark 0.0.2. But the vocabulary of mathematics is not like vocabulary of foreign language; these will not be new words for old ideas; these will be new words for new ideas.

Remark 0.0.3. Just as it will all take us many years to learn what love is, just as we will have to update our understanding as time passes, and just as this conceptualization will only change fruitfully as you invest time in this idea of love, your idea of the word "space" will also require both the passage and investment of time to develop. Be patient with yourself.

0.1 A moment of confusion; our goals

Let me open the floodgates for a moment to lay on you some definitions. You may see terms you are not familiar with, and what happens in the next five minutes, you are not responsible for knowing just yet.

Definition 0.1.1. A *topological space* is the data of a pair

$$(X, \mathcal{T})$$

where X is a set, and $\mathcal{T}$ is a collection of subsets of X, satisfying the following conditions:

1. The empty set $\emptyset$ and X itself are in $\mathcal{T}$,
2. For any finite collection $U_1, \ldots, U_n$ in $\mathcal{T}$, the intersection $U_1 \cap \ldots \cap U_n$ is in $\mathcal{T}$, and
3. For *any* collection $\{U_\alpha\} \subset \mathcal{T}$, the union $\bigcup_\alpha U_\alpha$ is in $\mathcal{T}$.

Definition 0.1.2. Let $(X, \mathcal{T})$ be a topological space. An element $U \in \mathcal{T}$ is called an *open set* of X.

Definition 0.1.3. Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be two topological spaces. A function $f : X \to X'$ is called *continuous* if for any open set $U' \in \mathcal{T}'$, the preimage $f^{-1}(U')$ is an open set of X.

That took a few minutes. Believe it or not, if you understand the above three definitions, you have completed at least half the class.

But you do not understand, at least at this moment. A major *goal* of this class will be to understand the above definitions, and as I said, this will take time. We have the semester for a reason.

Another major goal of this class will be for you to begin thinking the way mathematicians do. This means to understand how to *come* to an understanding.

0.2 Continuity

So let's get to it.

Here's an often unspoken tip about being a mathematician: Oftentimes, we give definitions of objects, only to be able to understand the *functions* between them.

Example 0.2.1. For now, you can pretend that I only gave the definition of a topological space so that I can tell you what a continuous function is.

At this point I want you to feel funny: You already know what a continuous function is! (Or you're supposed to, at least.) Let's review.

Let us fix a function

$$f : \mathbb{R} \to \mathbb{R}.$$

I will take some time to dissect this notation for you.

1. Here, $\mathbb{R}$ is the collection of all real numbers. It is a set. It contains things like 0, 1, 2, 3, π, $-\pi$, e, $3/4$, $\sqrt{2}$, and so forth.

2. The colon :, along with the arrow $\to$, indicates that I am defining a function. This function has domain $\mathbb{R}$, and target $\mathbb{R}$ as well. In plain English, this means I am defining an assignment which eats a real number, and spits out a (possibly different) real number. An example would be something that takes a real number and outputs its square; this is often referred to as the function $f(x) = x^2$.

3. The letter f indicates the *name* I want to give to the function. For example, if I were to write "$g : \mathbb{R} \to \mathbb{R}$," I am merely declaring that from hereon, I will be talking about a function called g.

4. I used the phrase "let us fix a function f." This is jargon, the same way lawyers use legal terms, mathematicians use their own linguistic conventions. "Let us fix a function" does not mean that we all choose our favorite function. "Let us fix a function," in fact, means almost the opposite—it means that we are about to discuss something that is true for an *arbitrary* function. You are allowed to have a function in mind, but you must also be aware that the devil may be in the room, and the devil may choose a completely different (and horrible-looking) function.

Discussion 0.2.2. What does it mean for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous?

(Some discussions will talk about intuitive meanings, which is fine. Some discussions may be imprecise. Some will get at a definition.)

In this discussion, I expect some ideas to come up. Things like:

1. The graph of f has no "jumps."

2. The graph of f is "connected."

3. The graph of f "divides" the plane into two halves.

4. If a sequence x_n converges to x, then the sequence $f(x_n)$ converges to $f(x)$.

5. f satisfies the "epsilon-delta" definition.

My expectation is that almost everybody will have some intuition—some correct, some incorrect–about what a continuous function is. I suspect only a few people will have remembered what the definition that you learn in calculus is:

Definition 0.2.3. A function $f : \mathbb{R} \to \mathbb{R}$ is called *continuous* if and only if:

For every x and for every $\epsilon > 0$,
there exists a $\delta > 0$ so that
for every x', we have

$$|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon.$$

At this point, you have seen a "definition" of continuity (Definition 0.2.3) and you have also discussed your *intuition* of continuous functions. You should notice that the definition and the intuition may look very different.

So, we have already seen one of the major ideas of modern mathematics, and of this class: Continuity. We will talk more about this as time goes on, of course. For next time, all you need to do is explore this idea on your own terms, and turn in the result.

0.3 Getting to know $\mathbb{R}$

As my preview might have hinted, we need to understand and define what it means for a function to be continuous even when the domain and codomain may not be $\mathbb{R}$. To do that, we will now examine what enabled us to define a notion of continuity for functions from $\mathbb{R}$ to $\mathbb{R}$; understanding the ingredients in our familiar case will allow us to extend our ideas to the unfamiliar cases.

Remark 0.3.1. You have known $\mathbb{R}$—the set of real numbers—for a long time. But like a family member you have known a long time, sometimes it is only with intense reflection that you realize the things you have taken for granted.

You are, believe it or not, very familiar with $\mathbb{R}$—like family. But what are we relying on to define continuity?

Discussion 0.3.2. What *properties* or *structures* of $\mathbb{R}$ are we using in the epsilon-delta definition of continuity (Definition 0.2.3)?

Some things you may come up with:

1. We know how to "subtract" elements when we write things like $x - x'$ or $f(x) - f(x')$.

2. We know how to take absolute value when we write something like $|x - x'|$ (or $|f(x) - f(x')|$.

3. We know how to *compare* $|x - x'|$ with ϵ so we can write something like ϵ.

4. In fact, we also have intuition about the first two things Hiro listed: "subtracting" and "take absolute value" combine to give us a notion of "distance" between two points—$|x - x'|$ is the distance from x to x'.

I would now like to focus on this idea of *distance*. This will lead us to one of the most intuitive ways to talk about spaces and continuous maps between them.

Metric spaces

As I've mentioned before, we will be very much interested in notions of distance. This is because—at least based on our everyday experiences—whenever we think of a shape or a space, we can certainly measure the distance between two points on that shape or space. Moreover, we saw in the previous section that the very definition of continuity (for functions $f : \mathbb{R} \to \mathbb{R}$) utilized the notion of distance.

Remark 0.3.3. Later in our course, we will study shapes where it is *unnatural* to speak of distances; this may come as a surprise, but more on that later.

Today, we'll isolate what properties of "distance" are reasonable to expect on the kinds of shapes we're familiar with.

0.4 Preliminaries: Products

First, let me remind you of some background; it's okay if this is the first time you've seen these background ideas.

Definition 0.4.1. Let X and Y be two sets. Then the notation

$$X \times Y$$

represents their product; this is also sometimes called the *Cartesian product* of X and Y.

$X \times Y$ is a set whose elements are ordered pairs

$$(x, y)$$

with $x \in X$ and $y \in Y$.

Example 0.4.2. Let X be a set of three people named Alejandra, Bill, and Candace. Let Y be a set of two people named Seungwan and Theo. Then $X \times Y$ has exactly six elements, and they are listed as follows:

- (Alejandra, Seungwan)
- (Bill, Seungwan)
- (Candace, Seungwan)
- (Alejandra, Theo)
- (Bill, Theo)
- (Candace, Theo)

Note that (Theo, Candace) is *not* an element of $X \times Y$. This is what the word "ordered" means in "ordered pair."

Example 0.4.3. X and Y may be the same set. For example, let $\mathbb{R}$ be the set of all real numbers, and set $X = Y = \mathbb{R}$. Then $X \times Y$ has another names, called $\mathbb{R}^2$.[1]

We will often denote an element of $\mathbb{R}^2$ by (x_1, x_2).

Example 0.4.4 (Iterated products)**.** You can iterate the product construction. For example, if you have three sets X and Y and Z, it makes sense to form the sets

$$(X \times Y) \times Z \qquad \text{and} \qquad X \times (Y \times Z).$$

These two sets are *not* the same, but there is a natural bijection between them. This distinction need not worry you for the time being, but thinking through this statement carefully will do you a lot of good in the future.

There is yet another set you can construct, which we will write

$$X \times Y \times Z.$$

The elements of $X \times Y \times Z$ consist of ordered triplets (x, y, z) where $x \in X, y \in Y$ and $z \in Z$.

Of course if you have a collection of sets, you can take the product of all of them.

[1] This explains the notation $\mathbb{R}^2$; it is quite informal and lazy, but the rationale behind the notation is the suggestive equality $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Example 0.4.5 (Euclidean space)**.** An important example is $\mathbb{R}^n$, which is the n-fold Cartesian product of $\mathbb{R}$. You may be more familiar thinking of $\mathbb{R}^n$ as n-dimensional Euclidean space.

Example 0.4.6. Fix a set X. We will soon think about functions

$$X \times X \to \mathbb{R}.$$

This means that, for every ordered pair of elements (x_1, x_2) with $x_1, x_2 \in X$, we will assign a real number.

When $X = \mathbb{R}$, you have seen many examples of such functions:

1. Addition, which sends a pair (x_1, x_2) to $x_1 + x_2$.
2. Subtraction, which sends (x_1, x_2) to $x_1 - x_2$.
3. Multiplication, which sends a pair (x_1, x_2) to the product $x_1 \cdot x_2$.
4. Division is *not* an example of a function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. While you may happily write a formula taking the pair (x_1, x_2) to the quotient x_1/x_2, this is not defined when $x_2 = 0$.
5. The distance function, which takes a pair (x_1, x_2) to the distance between them: $|x_2 - x_1|$.

0.5 Definition of metric spaces

Notation 0.5.1. Let X be a set. We will often write an element of $X \times X$ as (x, x'). (In the previous section, we used the notation (x_1, x_2) instead.) The symbol x' is read "x prime." The reason for this is that we will soon let $X = \mathbb{R}^n$, so that X itself is made up of ordered tuples; the dual roles of subscripts will then become quite confusing, so we will use the "prime" symbol.

Example 0.5.2 (Distance on $\mathbb{R}^2$)**.** We've already talked about the distance function on the set $X = \mathbb{R}$:

$$d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x, x') \mapsto |x' - x|.$$

Let's now think about $X = \mathbb{R}^2$. Given two points in $\mathbb{R}^2$, what is the distance between them?

The Pythagorean theorem tells us: Given two points $x = (x_1, x_2)$ and $x' = (x'_1, x'_2)$ in $\mathbb{R}^2$, the length of the path between them is given by

$$d(x, x') = \sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2}. \tag{0.5.0.1}$$

Note that this function has a lot of intuitive properties:

1. If $x = x'$, then the distance between x and x' is zero.
2. Conversely, if the distance between two points is zero, they are equal points.
3. The triangle inequality: This is not always intuitive for most students, but it is a fact of life. If you have three points x, x', x'', then the distance from x to x'' is *at most* the sum of the distances between x and x', and between x' and x''. (You should draw a picture.)
4. Symmetry: The distance from x to x' is the same as the distance from x' to x.

There are others, but we will leave that for exercises or personal exploration.

Example 0.5.3. Now let's consider a different shape X. For example, let's take X to be any *arbitrary* subset of $\mathbb{R}^2$.

Is there still a notion of distance between two points of X? Yes; you could just measure the distance as you normally would inside $\mathbb{R}^2$. Thus we have a function

$$d : X \times X \to \mathbb{R}$$

by the exact same formula in (0.5.0.1).

Does this function satisfy all the properties we talked about in Example 0.5.2? Yes.

We isolate these properties to give the following definition:

Definition 0.5.4. A *metric space* is the data of a pair (X, d) where X is a set, and

$$d : X \times X \to \mathbb{R}$$

is a function satisfying the following properties:

(0) $d(x, x') = 0 \iff x = x'$.

(1) (Symmetry) $d(x, x') = d(x', x)$.

(2) (Triangle inequality) $d(x, x') + d(x', x'') \geq d(x, x'')$.

Remark 0.5.5. Intuitively, a metric space is a set with some notion of distance between two points. Note that a single set X may admit many different examples of a function d. When should we consider to metric spaces to be equivalent? We will get to that in Section **??**.

Exercise 0.5.6. Show that if (X, d) is a metric space, then for any pair $x, x' \in X$, we have that $d(x, x') \geq 0$.

At this point, what questions do you have?

0.6 Continuous maps

Definition 0.6.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Fix a function $f : X \to Y$. We say that f is *continuous* if:

For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')).$$

Remark 0.6.2. Informally, the above definition says that a continuous function between metric spaces is one that respects the idea of *closeness*.

You should think of ϵ as a number a mortal enemy gives you, daring you to be ϵ-close (i.e., within ϵ) of $f(x)$.

You should think of δ as the number that allows you to vanquish that dare: If x' is any element δ-close to x, then you know that $f(x')$ is ϵ-close to $f(x)$.

0.7 Examples of metric spaces

These are all useful examples. You should do your best to understand them.

Example 0.7.1 (Euclidean space)**.** Let $X = \mathbb{R}^n$. Define

$$d(x, x') = \sqrt{\sum_{i=1}^{n} (x'_i - x_i)^2}.$$

This is called the "standard" metric on $\mathbb{R}^n$.

Example 0.7.2 (Subsets of metric spaces)**.** Let (Y, d_Y) be a metric space and fix a subset $X \subset Y$. Then for any $x, x' \in X$, define $d(x, x') = d_Y(x, x')$. This renders (X, d) a metric space.

Example 0.7.3 (Subsets of Euclidean space)**.** In particular, if (Y, d_Y) is Euclidean n-dimensional space with the standard metric, we see that any subset of Euclidean space inherits a metric space structure.

Example 0.7.4 (Continuous functions on a closed, bounded interval)**.** Here is one of the more "infinite-dimensional" and difficult-to-visualize examples.

Let X be the set of continuous functions $f : [0, 1] \to \mathbb{R}$ on the closed interval from 0 to 1. Given two functions f and g, define

$$d(f, g) = \int_0^1 |f - g| dt.$$

That is, the distance from f to g is defined to be the integral of $|f - g|$ over the interval from 0 to 1.

Example 0.7.5 (Sequences with finite support)**.** Let X the set of all sequences of real numbers such that the sequence is non-zero for only finitely many entires. Given two sequences $x = (x_1, x_2, \ldots)$ and $x' = (x'_1, x'_2, \ldots)$, we define

$$d(x, x') = \sum_{i=1}^{\infty} |x'_i - x_i|.$$

Note that the summation converges precisely because the summands are non-zero for only finitely many i.

It is also incredibly useful to consider "trivial," or maybe the "simplest," or most "cheating" examples when you see a new definition.

Example 0.7.6. Let X be the empty set. Then there is a unique function $d : X \times X \to \mathbb{R}$, and this renders X a metric space. (It is okay if you want to ignore this example.)

Example 0.7.7. Let X be any set. Then define d as follows:

$$d(x, x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x'. \end{cases}$$

This makes (X, d) into a metric space. Informally, you can think of X as a weird metric space in which any two non-equal points are exactly distance

1 apart. It may be fun to imagine what such a thing looks like, especially when there are infinitely many points in X.

Note that the number 1 is not special here; it could be replaced by any positive number.

Last time:

Defn A fxn $f: \mathbb{R} \longrightarrow \mathbb{R}$ is called continuous, or C^0, if

$$\forall x \in \mathbb{R},\ \forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t.}$$

$$d(x', x) < \delta \implies d(f(x'), f(x)) < \varepsilon.$$

Defn A metric space is a set X together w/ a function

$$d : X \times X \longrightarrow \mathbb{R}$$

such that

(0) $d(x, x') = 0 \iff x = x'$

(1) $d(x, x') = d(x', x)$

(2) $d(x, x') + d(x', x'') \geq d(x, x'')$.

Defn Let (X, d_X) and (Y, d_Y) be metric spaces. Then a function
$$f : X \longrightarrow Y$$
is called continuous, or C^0, if

$$\forall\, x \in X,\ \ \forall\, \varepsilon > 0,\ \ \exists\, \delta > 0 \text{ s.t.}$$

$$d_X(x', x) < \delta \Rightarrow d_Y(f(x'), f(x)) < \varepsilon.$$

Rmk I motivated the idea of a metric space by asking: what do we need to define continuity? The ε-δ definition suggested all we need is a notion of distance. That's what the function d captures.

テキスト

At this point, what should you demand of me?
(As with anybody else who gives you a new definition.)

- Examples
- Motivation (i.e., why is this useful?)

Before we move on, let me mention:

<u>Def'n</u> Let (X,d) be a metric space.
Then d is called a <u>metric</u> on X.

The standard metric

Example Let $X = \mathbb{R}$, and

$$d: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(x, x') \longmapsto |x' - x|.$$

Exer Verify this example is indeed a metric space.

Example Let $X = \mathbb{R}^n$, and

$$d: X \times X \longrightarrow \mathbb{R}$$
$$(x, x') \longmapsto \sqrt{\sum_{i=1}^{n} (x_i' - x_i)^2}.$$

Exer Show this is a metric space.

Solution: Check all three conditions:

(R) (0) If $d(x,x')=0$, then $|x'-x|=0$.

- i.e., $x'-x=0$. So $x=x'$.

If $x=x'$, then $d(x,x')=|x-x|=|0|=0$.

(1) $d(x,x')=|x'-x|$

$=|x-x'|$ (b/c $|x-x'| = |-(x'-x)| = |x'-x|$)

$=d(x',x)$

(2) $d(x,x')+d(x',x'')=|x'-x|+|x''-x'|$

$\geq |(x'-x)+(x''-x')|$ ☆

$=|x''-x'+x'-x|$

$=|x''-x|$

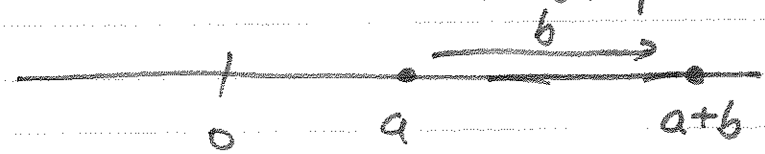
$=d(x,x'')$.

<u>Rmk</u> Make sure you understand $(\ast)$. For any pair of real numbers (a,b), we have

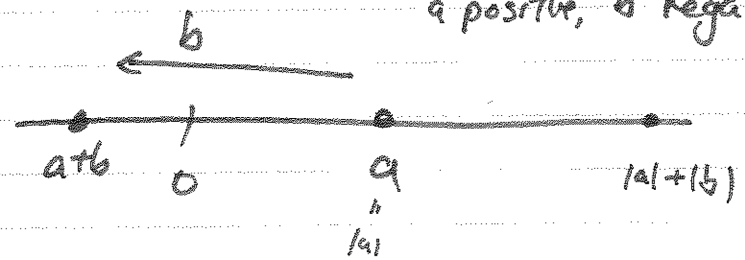
$$|a+b| \leq |a| + |b|.$$

(If a and b have same sign, then $|a+b| = |a| + |b|$. Otherwise, $|a+b| < |a| + |b|$.)

a, b both positive.



a positive, b negative.



We'll verify that $\sqrt{\sum (x_i' - x_i)^2}$ is a metric another time; for now let's see other examples.

The discrete metric

One of the more surprising facts is that $\mathbb{R}^n$ has many different metric space structures you can put on it.

Ex (The discrete metric space structure).

Define

$$d(x,x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x'. \end{cases} \qquad (\ast\ast)$$

Exer Show this makes $\mathbb{R}^n$ into a metric space. What did you use about $\mathbb{R}^n$?

Sol'n:

(0) $x = x' \Longrightarrow d(x,x') = 0$ by def'n.

$d(x,x') = 0 \Longrightarrow d(x,x') \neq 1$

$\Longrightarrow x = x'$ by def'n.

(1) If $x = x'$, $d(x,x') = 0 = d(x',x)$.

If $x \neq x'$, $d(x,x') = 1 = d(x',x)$.

(2) $$d(x,x') + d(x',x'') = \begin{cases} 1+1 & x \neq x' \text{ and } x' \neq x'' \\ 1+0 & \\ 0+1 & \\ 0+0 & x = x' = x'' \end{cases}$$

while

$$d(x,x'') = \begin{cases} 1 & x \neq x'' \\ 0 & x = x'' \end{cases}.$$

//

We used <u>nothing</u> about $\mathbb{R}^n$.

That is, we observe:

Ex Let X be any set. Then $(\star\star)$ defines a metric space structure on X.

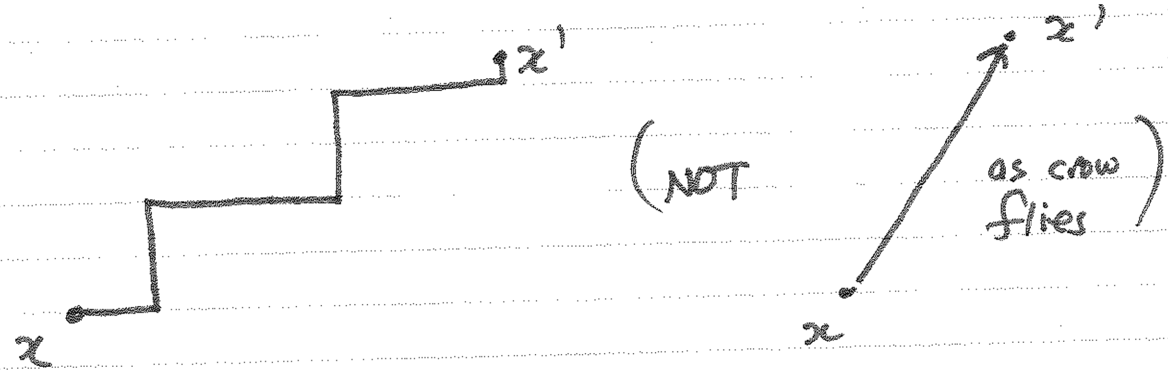
Rmk In fact, the number 1 is irrelevant — for any number $C > 0$, the assignment

$$d(x,x') = \begin{cases} 0 & x = x' \\ C & x \neq x' \end{cases}$$

is a metric.

Manhattan metric

If you lived in a city w/ a grid system (e.g., Manhattan), you wouldn't be able to get from x to x' as the crow flies; you'd need to walk along blocks:



what's the distance you've traveled if you walk along the grid?

$$|x_1' - x_1| + |x_2' - x_2|.$$

Exer Show

$$d(x, x') = \sum_{i=1}^{n} |x_i' - x_i|$$

is a metric on $\mathbb{R}^n$.

<u>Sol'n:</u> (0) $x = x' \Longrightarrow |x_i' - x_i| = |0| = 0$

$\forall$ i, so

$$\Longrightarrow \sum_{i=1}^{n} |x_i' - x_i|$$

$$= \sum_{i=1}^{n} 0$$

$$= 0.$$

On the other hand, if $d(x,x') = 0$, then (because $|\cdot| \geq 0$) it must be that $\forall i$, $|x_i' - x_i| = 0$. Hence $x = x'$.

(1) $|x_i' - x_i| = |-(x_i' - x_i)| = |x_i - x_i'|$ so

$d(x', x) = d(x, x')$.

(2) $d(x,x') + d(x',x'') = \sum |x_i' - x_i| + \sum |x_i'' - x_i'|$.

For all i, we have:

$$|x_i' - x_i| + |x_i'' - x_i'| \geq |x_i'' - x_i' + (x_i' - x_i)|$$

$$= |x_i'' - x_i|$$

Hence

$$d(x,x')+d(x',x'') \geq \sum_{i=1}^{n} |x_i''-x_i|$$

$$= d(x,x''). \quad //$$

<u>Rmk</u> The Manhattan metric is sometimes called the taxicab metric.

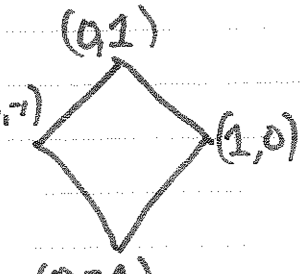
<u>Exer</u> Draw the following subsets of $\mathbb{R}^2$:

(a) $\{x \text{ s.t. } d(x,0)=1\}$

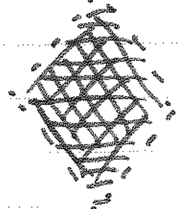
(b) $\{x \text{ s.t. } d(x,0)<1\}$.

(using the taxicab metric).

<u>Sol'n</u> (a)



(b)



The l^∞ metric

(pronounced "ell infinity")

Let

$$d(x,x') = \max_{i=1,\cdots,n} |x_i' - x_i|.$$

That is, we declare the distance from x to x' to be the largest of the distance between their coordinates.

Ex $x = (2,2,2)$ $x' = (3,4,5)$

then $|x_1' - x_1| = 1$ $(3-2)$

$|x_2' - x_2| = |4-2| = 2$

$|x_3' - x_3| = |5-2| = 3.$

$$\max_i |x_i' - x_i| = \max\{1,2,3\} = 3.$$

So $d(x',x) = 3.$

Exer Show this is a metric on $\mathbb{R}^n$.

Soln: (0) $x' = x \Rightarrow \forall i, \ |x_i' - x_i| = 0$

$$\Rightarrow \max_i |x_i' - x_i| = 0.$$

If $\max_i |x_i' - x_i| = 0$, then $|x_i' - x_i| = 0$ for all i.

Hence $x_i = x_i' \ \forall i$, hence $x' = x$.

(1) $d(x', x) = d(x, x')$ is straightforward.

(2) For all i, we have

$$|x_i'' - x_i| \leq |x_i'' - x_i'| + |x_i' - x_i|$$

$$\leq \max_{j=1,\dots,n} |x_j'' - x_j'| + \max_{k=1,\dots,n} |x_k' - x_k|$$

$$= d(x'', x') + d(x, x').$$

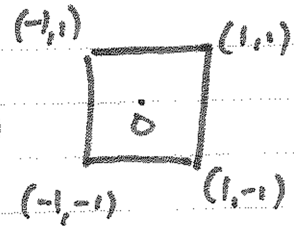
Hence $d(x, x'') = \max_i |x_i'' - x_i| \leq d(x'', x') + d(x, x')$. //

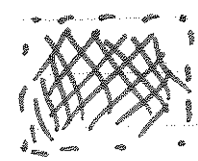
Exer Draw

(a) $\{x \text{ s.t. } d(x,0) = 1\}$

(b) $\{x \text{ s.t. } d(x,0) < 1\}$

using ℓ^∞ metric.

Soln: (a) 

(b) 

__Summary__: We've seen __four__ metrics on $\mathbb{R}^n$!

__Notation__

$$d_{std}(x,x') = \sqrt{\sum_{i=1}^{n} (x_i' - x_i)^2}$$

(standard metric)

$$d_{taxi}(x,x') = \sum_{i=1}^{n} |x_i' - x_i|$$

(taxicab metric)

$$d_{\ell^\infty}(x,x') = \max_{i=1,\dots,n} |x_i' - x_i|$$

(ℓ^∞ metric)

$$d_{discrete}(x,x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x' \end{cases}$$

(discrete metric)

Project for today

Let $X = Y = \mathbb{R}^2$, and let

$$f : X \longrightarrow Y$$

be the identity function. (That is, $f(x) = x$.)

For which metrics d_X and d_Y on X and Y (from the four metrics we've discussed) is f continuous?

⚠ When you do NOT yet have an intuition, definitions are all you've got to go on.

(Today, you probably have very little intuition for these new metrics!)

Lecture 3

3.1 Remarks on writing assignment

I'd like to touch on a few themes that came up in your writing assignments

3.1.1 Continuity of *objects* and of *functions*

Many people thought of continuity in terms of properties of an object—for example, they explored in what sense $\mathbb{R}$ seemed like a continuous object.

As I mentioned in class, we define certain ideas (like topological spaces) to be able to speak of certain functions between them (like continuous functions). So it is indeed a good investment to think about what it means for an object like $\mathbb{R}$ to have certain properties that allow us to speak of continuous functions out of, or to, $\mathbb{R}$.

3.1.2 f is continuous if f is defined and...

Many students learn in calculus that f is continuous at a if three conditions are satisfied:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists, and
3. this limit equals $f(a)$.

However, that first condition is superfluous when you have already declared that $f : \mathbb{R} \to \mathbb{R}$ is a function from $\mathbb{R}$ to $\mathbb{R}$. That the domain is $\mathbb{R}$ (and in particular, *all* of $\mathbb{R}$!) means that f is defined at any $a \in \mathbb{R}$.

The reason that you were taught the above definition is because most calculus educators are also taught things that way, and the reason they are taught things that way is because they are not expected to teach with the precision and sophistication that should be expected of mathematicians (and of math majors, at least). For example, your calculus class probably did not consist only of future math majors, so that level of precision was dropped. If one has a function $f : X \to Y$, in particular, f is defined on every element of X.

Finally, in practice, one sometimes writes a formula, and one needs to verify that f is indeed a function on all of $\mathbb{R}$. For example, if I declare

$$f(x) = \frac{\sin x}{x}$$

it is not obvious at first whether f is defined at $x = 0$. Thus you may not know whether to declare f to be a function on all of $\mathbb{R}$, or only on the set $\mathbb{R} \setminus \{0\}$. So this is at least the beginnings of why so many people are careful about verifying that f is defined somewhere—we often write formulas, but formulas do not always make sense everywhere.

3.1.3 Smooth versus continuous

Some people were confused about the existence of "kinks" or "non-smooth" phenomena, and said that part of their intuition of continuity contained a "smoothness" requirement. I want to dissuade you from thinking about smoothness.

First, "smooth" has a technical meaning in math, as it turns out. A function is smooth if you can take a derivative as many times as you want. And while every function is smooth, not every continuous function is smooth. So keep that in mind.

Next, consider the example of $f(x) = |x|$. The graph of this function is not a "smooth" object, as it clearly has a kink, or a corner, at the origin. But the function is still continuous.

3.1.4 "Approached from either side."

Many people spoke of limits. They said that a limit $\lim_{x \to a} f(x)$ exists if "when approached from either side" the limiting value is equal.

This is a fine intuition for limits in $\mathbb{R}$, but what if you are in $\mathbb{R}^2$? If you chose a point $a \in \mathbb{R}^2$, there are many ways to "approach it"–not just in terms of directions, but also in terms of the shape of the path that you take to get to a. (For example, one could spiral toward a.) You see the situation can be even more complicated in $\mathbb{R}^n$ for high n.

The ϵ-δ definition of continuity, which we gave without discussing the notion of limit, ignores any "choice" of direction or path by which you approach a. It simply says that if you want to guarantee that f attains values close to the value $f(a)$, you simply need to be close to a itself.

3.1.5 Picturing

Many people said they had a hard time picturing the definition of continuity.

That's perfectly normal. Indeed, even the most seasoned mathematicians probably do not imagine the most general and crazy examples of continuity; they may simply imagine something like the function $f(x) = |x|$.

This is also the *power* of abstract definitions; in life we sometimes have to prove or understand things without being able to visualize them.

3.1.6 Understanding

Many people said they do not understand the $\epsilon - \delta$ definition. That is normal.

Let me share a quote from John von Neumann.

"In mathematics you don't understand things. You just get used to them."

I strongly disagree with this quote, but it rests on what you mean by understanding. (Let's not get into that discussion.) I give this quote note as a model, but as comfort; even seasoned mathematicians feel like they do not understand things.

An analogy I dislike, but will use anyway because it is helpful, is the following: You may not understand how a car works, but you can still drive it. Rest assured that most mathematicians do not understand everything they use; and at some point, they have simply had to drive a car without taking apart every component.

3.1.7 What's the use of continuity?

This is a great question.

Let me give some sample applications:

Theorem 3.1.1. Let $[a, b]$ be a closed, bounded interval. Fix a function $f : [a, b] \to \mathbb{R}$. If f is continuous, then f attains a maximum and a minimum.

This is a very powerful theorem.

As a non-example, consider $\tan x$, which does not attain a minimum or a maximum—this shows the necessity of the interval of definition being closed.

In this class, we will generalize the above theorem to any continuous function whose domain is compact. Compactness is a useful notion that comes up over and over in mathematics—it is useful because it identifies a large class of spaces that behave well and are easily controlled; it is also a condition that is easy to check in many cases.

3.2 Some facts about metrics

Now let's get back to metric spaces.

Exercise 3.2.1. Let (X, d) be a metric space. Prove that for any $x, x' \in X$,

$$d(x, x') \geq 0.$$

Proof. Combining the triangle inequality and property zero, we have

$$d(x, x') + d(x', x) \geq d(x, x) = 0.$$

By symmetry, we have

$$2d(x, x') \geq 0.$$

Dividing by 2, we are done. □

The above exercise shows that the notion of distance in a metric space fits our physical intuition that the distance between any two points ought to be non-negative. (And, by property zero, positive when $x \neq x'$.)

Last time we left off as we were about to study the continuity of the identity function between various metric space structures on $\mathbb{R}^n$.

Of course, there were four different metric spaces. That means there are a total of

$$4 \times 4 = 16$$

different combinations of metric for which we would have to verify continuity. That's a lot. But here's a useful fact that will cut down that number:

Exercise 3.2.2. Fix three metric spaces

$$(X, d_X), \qquad (Y, d_Y), \qquad (Z, d_Z)$$

and two functions $f : X \to Y$ and $g : Y \to Z$.

Show that if both f and g are continuous, then so is the composition $g \circ f$.

Proof. We must verify the following:

For all $x \in X$, and for all $\epsilon > 0$, there exists δ such that

$$d_X(x, x') < \delta \implies d_Z(gf(x), gf(x')) < \epsilon.$$

Well, by the continuity of g, we know that for all y (and in particular, for $y = f(x)$) and for all ϵ, there exists some ϵ' so that

$$d_Y(f(x), y') < \epsilon' \implies d_Z(gf(x), g(y')). \tag{3.2.0.1}$$

And by the continuity of f, we know that for all x, and all ϵ', there exists δ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon'. \tag{3.2.0.2}$$

Thus, given ϵ, choose ϵ' satisfying (3.2.0.1), then choose δ satisfying (3.2.0.2). We are finished. □

3.2.1 Our favorite metrics on $\mathbb{R}^n$

Last time we defined

1. The standard metric

$$d_{std}(x, x') = \sqrt{\sum_{i=1}^{n} (x'_i - x_i)^2}$$

2. The discrete metric

$$d_{discrete}(x, x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x' \end{cases}$$

3. The taxicab metric

$$d_{taxi}(x, x') = \sum_{i=1}^{n} |x'_i - x_i|$$

4. The l^∞ metric
$$d_{l^\infty}(x,x') = \max_{i=1,\ldots,n} |x'_i - x_i|.$$

We have the identity function

$$\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n, \qquad x \mapsto x.$$

We can put any of the metric space structures above on $\mathbb{R}^n$—for which choices is the identity function continuous?

Example 3.2.3. Consider

$$\mathrm{id} : (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{discrete}).$$

Is this a continuous function?

I want you now to get into groups and investigate.

Upshot: You should find that the above example is the only example for which the identity function is not continuous.

3.2.2 A tip

How do you prove a function is continuous? In practice, it comes down to understanding what the condition

$$\text{If } x' \text{ is such that } d_X(x,x') < \delta, \text{ then } d_Y(f(x), f(x')) < \epsilon$$

means. To understand that such x' "look like," we first try to understand what $f(x')$ might look like.

What does $d_Y(f(x), f(x')) < \epsilon$ mean? That is, what does it mean for $f(x')$ to be distance less than ϵ apart from $f(x)$? It means—tautologically—that $f(x')$ is in the following set:

$$y' \text{ such that } d_Y(f(x), y') < \epsilon.$$

This is what one calls the *open ball of radius ϵ centered at $f(x)$* (with respect to the metric d_Y).

Example 3.2.4. We have drawn examples of open balls of radius 1 for our various metrics on $\mathbb{R}^n$.

A common strategy to prove the continuity of a function $f : X \to Y$ is to understand the set of those x' that end up in the open ball of radius ϵ centered at $f(x)$. That is, under what circumstances do we have that

$$f(x') \in \{\ y' \text{ such that } d_Y(f(x), y') < \epsilon.\ \}?$$

This comes down to understanding the following:

$$\text{What } x' \text{ are contained in the preimage } f^{-1}(\{\ y' \text{ such that } d_Y(f(x), y') < \epsilon.\ \})?$$

In other words,

$$\text{What is} f^{-1}(\{\ y' \text{ such that } d_Y(f(x), y') < \epsilon.\ \})? \tag{3.2.2.1}$$

Note that x is always contained in this set.

Once you understand this set, you can ask the following: Is there an open ball centered at x (of some radius $\delta > 0$) contained in this set?

If you can find such a δ, and if you can do this for every $\epsilon > 0$ and every $x \in X$, you have proven the continuity of $f : X \to Y$.

Let me summarize. The following statements are more or less equivalent descriptions of the process:

1. To show $f : X \to Y$ is continuous, you must answer in the affirmative: "Is there an open ball of radius $\delta > 0$ centered at x contained in the preimage of the open ball of radius ϵ centered at $f(x)$?" for every choice of $x \in X$ and for every choice of $\epsilon > 0$.

2. Continuity comes down to verifying that you can always find open balls centered at x contained in the preimage of open balls centered at $f(x)$.

We got back together and discussed. We discussed the following three examples:

- $\text{id} : (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{std})$.
- $(\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{taxi})$.
- $(\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{discrete})$.

3.2.3 The identity function from a metric space to itself

Proposition 3.2.5. $\mathrm{id} : (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{std})$ is continuous.

Remark 3.2.6. We are verifying that f is continuous for the case $X = Y = \mathbb{R}^n$, $d_X = d_Y = d_{std}$, and $f = \mathrm{id}$. The proof will thus use the notations X, Y, d_X, d_Y to make clear when I am speaking of the domain, or of the codomain.

For the proof, we follow the tip:

Proof of Proposition 3.2.5. Note that for any $x \in \mathbb{R}^n$, $f(x) = x$ because we have chosen $f = \mathrm{id}$.[1] Note also the following:

$$\begin{aligned} d_X(x, x') &= d_{std}(x, x') \\ &= d_{std}(f(x), f(x')) \\ &= d_Y(f(x), f(x')). \end{aligned} \tag{3.2.3.1}$$

That is,

$$d_X(x, x') = d_Y(f(x), f(x')). \tag{3.2.3.2}$$

As such, given any $\epsilon > 0$, let us simply set δ to be any positive real number less than or equal to ϵ. Then if $d_X(x, x') < \delta$, we have that $d_Y(f(x), f(x')) < \epsilon$. This shows f is continuous. □

Note that we did not use anything about $\mathbb{R}^n$ or d_{std} in the proof—indeed, the string of equalities uses the notation d_{std}, but the equalities hold for any choice of metric so long as $d_X = d_Y$. We conclude:

Proposition 3.2.7. Let $X = Y$ and $d_X = d_Y$, and let $f : X \to Y$ be the identity function. Then f is continous.

Proof. Note that (3.2.3.2) is true when $f = \mathrm{id}$ and $d_X = d_Y$. Then follow the rest of the proof of Proposition 3.2.5. □

[1] That is, we have chosen f to be the identity function.

3.2.4 Isometric embeddings and isometries

In fact, we have done something even better. We do not need X to equal Y, nor for d_X to equal d_Y. The proof of Proposition 3.2.5 relied only on the equality (3.2.3.2). This is a useful condition, so let's give it a name.

Definition 3.2.8. Let (X, d_X) and (Y, d_Y) be metric spaces. Fix a function $f : X \to Y$. We say that f is an *isometric embedding* if f preserves distances. That is,

$$d_X(x, x') = d_Y(f(x), f(x'))$$

for all $x, x' \in X$.

If f is further a bijection, we say that f is an *isometry.*

Proposition 3.2.9. Let (X, d_X) and (Y, d_Y) be metric spaces. Fix a function $f : X \to Y$. If f is an isometric embedding, then f is continuous. In particular, any isometry is continuous.

Proof. Follow the proof of Proposition 3.2.5 beginning with (3.2.3.2). □

3.2.5 From standard to taxi

Then we verified that the identity function from $\mathbb{R}^n$ with the standard metric to $\mathbb{R}^n$ with the taxicab metric is continuous:

Proposition 3.2.10. Let $(X, d_X) = (\mathbb{R}^n, d_{std})$ and $(Y, d_Y) = (\mathbb{R}^n, d_{taxi})$. Then the identity function

$$f = \mathrm{id} : X \to Y$$

is continuous.

Proof. Fix $x \in X$. We note that the open ball of radius ϵ centered at $f(x)$[2] is a diamond centered at $f(x)$, whose distance from $f(x)$ to any of the corners of the diamond is ϵ. Because $f = \mathrm{id}$, the preimage of this diamond is the diamond itself (now considered as a subset of X).

In $\mathbb{R}^2$, we drew a picture to see that any diamond with $\epsilon > 0$ "clearly" contains an open ball of radius $\delta > 0$ centered at x, so long as δ is small enough. For a picture and precise formula, see the scanned notes.

To prove for $\mathbb{R}^n$ for general n, one employs the formula from the scanned notes to verify indeed that one can find $\delta > 0$ small enough so that the continuity condition holds. □

[2] with respect to $d_Y = d_{taxi}$!)

3.2.6 From standard to discrete

Proposition 3.2.11. The function

$$\mathrm{id} : (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{discrete})$$

is not continuous.

Proof. It suffices to exhibit an $x \in X$ and $\epsilon > 0$ such that for any $\delta > 0$, there exists some x' such that

$$d_X(x, x') < \delta \text{ and } d_Y(f(x), f(x')) \geq \epsilon.$$

For this, choose ϵ to be any positive real number less than or equal to 1. We saw last time that the open ball of radius ϵ (with respect to $d_{discrete}$) centered at $f(x)$ is then a single point, given by $f(x)$ itself.

Because $f = \mathrm{id}$, the preimage of this set is the same set—that is, the preimage if the set $\{x\}$ consisting of a single point, called x.

Of course, for any $\delta > 0$, the open ball of radius δ centered at x can *not* be contained in this singleton set. $\square$

Lecture 4

4.1 Some confusions

Students have come in with some confusions, so let me discuss them.

1. The notation
$$(X, d_X)$$
does *not* refer to some point on $\mathbb{R}^2$. It is a pair of things, but it is not a pair of numbers. For example, X is a set—perhaps the set of bananas in this room—and d_X is a metric on this set.

2. When I write something like
$$f : X \to Y \tag{4.1.0.1}$$
I mean a function from X to Y, and I have called the function f. Now, when I have chosen two metric spaces, I will often write
$$f : (X, d_X) \to (Y, d_Y). \tag{4.1.0.2}$$
The two notations (4.1.0.1) and (4.1.0.2) mean the same thing. That is, the latter notation still encapsulates a function from X to Y. The reason I include the metrics d_X and d_Y in the notation is because it is often important to remind the reader which metrics we are considering.[1]

3. Make sure you understand my notation using $x \in \mathbb{R}^n$ and the indices x_i. A point $x \in \mathbb{R}^n$ is determined by a finite collection
$$(x_1, x_2, \ldots, x_n)$$

[1] As we have seen, the same set may allow many different metrics.

of real numbers. Likewise, a point $x' \in \mathbb{R}^n$ is determined by a collection

$$(x'_1, x'_2, \ldots, x'_n)$$

of real numbers. So when I write a formula like

$$d_{l^\infty}(x, x') = \max_{i=1,\ldots,n} |x'_i - x_i|$$

I mean that I consider the collection of real numbers[2]

$$\{|x'_1 - x_1|, |x'_2 - x_2|, \ldots, |x'_n - x_n|\}$$

and I take the maximum number in this set (consisting of n real numbers). As an example, if $n = 4$ and I take two points

$$x = (\pi, \sqrt{2}, 1, 1)$$

and

$$x' = (8, 2, \sqrt{3}, 1)$$

then we would have

$$d_{l^\infty}(x, x') = \max\{|8 - \pi|, |2 - \sqrt{2}|, |\sqrt{3} - 1|, |1 - 1|\}$$

so

$$d_{l^\infty}(x, x') = 8 - \pi.$$

4.2 Last time

Last time we talked about the continuity of the following functions:

1. $\mathrm{id} : (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{std})$.
2. $\mathrm{id} : (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{taxi})$.
3. $\mathrm{id} : (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{discrete})$.

The main toolkit I wanted you to come away with was the following:

To investigate the continuity of a function $f : X \to Y$, you have to see if you can fit some open ball (of radius $\delta > 0$) into the preimage of an open ball (of radius $\epsilon > 0$).[3]

[2] Again, the x'_i are the coordinates of the point x', and the x_i are the coordinates of the point x

[3] The former is centered at $x \in X$, the latter is centered at $f(x) \in Y$.

4.3 Open sets of a metric space

4.3.1 Open balls

Today, I want to get us used to talk about open sets. Let's record as a formal definition some of the terms we've been using.

Definition 4.3.1. Let (X, d_X) be a metric space. Fix $x \in X$ and real a number $r > 0$.

The *open ball* of radius r, centered at x, is the set

$$\{x' \in X \text{ such that } d_X(x, x') < r.\}$$

We will use any of the following notations[4] to denote this set:

$$\text{Ball}(x; r) \qquad B(x; r) \qquad B_{d_X}(x; r) \qquad B_X(x; r).$$

Remark 4.3.2. Some authors extend the definition to the case $r = 0$. Then the open ball of radius 0 is the empty set; but we will not follow this convention.

4.3.2 Definition of open subset

Definition 4.3.3 (Open set of a metric space)**.** Let (X, d_X) be a metric space, and let $A \subset X$ be a subset.

We say that A is *open* if it can be written as the union of open balls.

4.3.3 Examples of open sets

Example 4.3.4. Let (X, d_X) be a metric space. Fix $x \in X$ and $r > 0$. Then $A = \text{Ball}(x; r)$ is an open set; it can be written as a union of a single open ball—namely $\text{Ball}(x; r)$.

The following is one of the more confusing examples for people.

Example 4.3.5. Let (X, d_X) be a metric space and let $A = \emptyset \subset X$ be the empty set.

Then A is open.

[4] Having all this notation is confusing, but you could already see why it was useful to have notation like d_X and d_Y to distinguish different metrics; these riffs/modifications to the notations also come in handy when disambiguating certain metric spaces.

Remark 4.3.6 (Digression into unions)**.** Let me explain this a little bit. When you think of unions of sets, you may think of something pictorial like a Venn diagram.

(Draw a Venn diagram.)

How do you take a union of sets? Well, you first specify the sets you want to take the union of, and then you combine their elements into a single set.

What if you specify no sets at all? Then the union of this collection of sets (the empty collection) is the empty set.

Some people are confused by Example 4.3.5 because they are only used to seeing unions of some non-zero number of sets.

But let me reiterate: A union of zero-many sets is the empty set. So the empty set can be written as the union of a collection (albeit an empty collection) of open sets.

Example 4.3.7. Let (X, d_X) be a metric space. Then the set $A = X$ itself is open.

To see this, fix any $r > 0$, and consider the collection

$$\{\mathrm{Ball}(x; r)\}_{x \in X}.$$

This is a collection of open balls. Let's explain the notation. The curly brackets $\{...\}$ means we are defining a set. The subscript $x \in X$ means for every $x \in X$, we can specify an element in this set. Which element? The notation $\mathrm{Ball}(x; r)$ means that the open ball $\mathrm{Ball}(x; r)$ is the element.

Confusingly, this is an example of a set of sets. You will get used to this.

Now, consider the union

$$\bigcup_{x \in X} \mathrm{Ball}(x; r).$$

This is a potentially gigantic union. There are many sets we are taking the union of—-for every $x \in X$, we are considering the open ball of radius r, and we are taking the union of every single one of these balls.

Note that this union is contained in X, as each ball is a subset of X. Moreover, any element of X is contained in the union, as any $x \in X$ is contained in the ball $\mathrm{Ball}(x, r)$. Thus,

$$X = \bigcup_{x \in X} \mathrm{Ball}(x; r).$$

So X is open (as it is written as a union of open balls).

Example 4.3.8. There is an even larger collection one can write down to prove that X is open. Consider the collection

$$\{\mathrm{Ball}(x,r)\}_{x\in X, r>0}$$

where now we are considering an open ball not just for every choice of $x \in X$, but also for every choice of real number $r > 0$.

Let me discuss a common confusion that this example can illustratively dispel—note that even if $r \neq r'$, the balls $\mathrm{Ball}(x,r)$ and $\mathrm{Ball}(x,r')$ may be the same. (We saw this in the discrete metric; for example, $r = 0.5$ and $r' = 0.4$ give the same open balls.)

Thus, the *subscripts* in the set notation do *not* need to uniquely specify an element of the set.

Another confusion: When exhibiting that a set A is open, you do not need to choose some efficient collection of open balls. For example, we have seen two ways to exhibit X as an open set. The second way we have seen (which not only takes an open ball for every x, but also for every $r > 0$) is far less "efficient" because we have so many open balls; that is fine. Do not be tempted to make a "snug" or "just right" collection of open balls to form a set, as overkill is sometimes useful.

Exercise 4.3.9. Let $X = \mathbb{R}^2$ and let A be the set

$$A = \{x \in \mathbb{R}^2 \text{ such that } d_{l^\infty}(0,x) < \delta\}.$$

This is the "open" square centered at the origin of width 2δ. (Put another way, this is the open ball of radius δ in $(\mathbb{R}^2, d_{l^\infty})$. For which of the following metrics on X is A an open set?

1. d_{std}
2. $d_{discrete}$
3. d_{taxi}
4. d_{l^∞}

Proof. All of them!

We must write A as the union of open balls. Thus, for every $x \in A$, we must exhibit some open ball contained in A and containing x.

Let us tackle d_{std} first. Given $x = (x_1, x_2)$, there is a well-defined (standard) distance to the boundary of A. Namely, consider the distances

$$|\delta - x_1|, \qquad |-\delta - x_1|, \qquad |\delta - x_2|, \qquad |-\delta - x_2|.$$

These measure the distance of x from the edges of the square. Note that because A is the open square[5], each of these distances is non-zero. Let r_x be the minimum of the four distances above. Then $\text{Ball}_{d_{std}}(x; r_x)$ is contained iN A. Thus we see that

$$\bigcup_{x \in A} \text{Ball}_{d_{std}}(x; r_x) = A.$$

This verifies that A is an open set in $(\mathbb{R}^2, d_{std})$.

I'll omit the proofs of the others. Note that d_{l^∞} is the "easiest" case because A is already an open ball in that case.

As for the other metrics, you simply need to find an r_x for every x such that the open ball of radius r_x (with respect to the chosen metric) is contained in A. For example, this is easy for the discrete metric—just choose r_x to be anything less than or equal to 1. For the taxicab metric, note that the diamond of corner-to-center length r fits inside the standard ball of radius r, so you could choose the same r_x as for d_{std}. $\square$

Exercise 4.3.10. Let $X = \mathbb{R}^2$ and let $A = \{x\}$ consist of a single point. For which of the following metrics on X is A an open set?

1. d_{std}

2. $d_{discrete}$

3. d_{taxi}

4. d_{l^∞}

Proof. Only for the discrete metric. Note that any open ball of positive radius in the other metrics contains at least two points (in fact, any open ball of positive radius contains infinitely many points in any of the non-discrete metrics); but A contains only one point, so A could not contain any open ball of positive radius. In particular, A cannot be written as the union of open balls. $\square$

[5]meaning the "boundary" of A is not part of A

Exercise 4.3.11. Let $X = \mathbb{R}$ and let $A = [-3, 3]$ be the closed interval from -3 to 3. For which of the following metrics on X is A an open set?

1. d_{std}
2. $d_{discrete}$
3. d_{taxi}
4. d_{l^∞}

Proof. Only the discrete metric.

To see why A is not open in the other metrics, note that if A can be written as a union of open balls, then in particular, there must be some open ball that contains $3 \in A$, and is contained in$A$.

But in any of the non-discrete metric, if an open ball B of positive radius contains 3, it must also contain some number larger than 3. But such a number is not contained in A. In particular, B could not be contained in A.

Finally, A is open in the discrete metric because we can write

$$A = \bigcup_{x \in A} \mathrm{Ball}_{d_{discrete}}(x; r)$$

for any $r \in (0, 1]$. $\square$

4.4 Open sets and continuity

In math, when you've found a way to translate one sophisticated thing into another, you've discovered something wonderful. We're about to discover something wonderful:

Theorem 4.4.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Fix $f : X \to Y$ a function. The following are equivalent:

1. f is continuous.
2. For any open set $V \subset Y$, the preimage $f^{-1}(V)$ is open.

4.4.1 Centering open balls

The proof will be streamlined if we utilize the following lemma:

Lemma 4.4.2. Let (X, d_X) be a metric space, and fix $x \in X$ and $r > 0$. Suppose that x' is contained in $\mathrm{Ball}_X(x, r)$. Then there exists $r' > 0$ such that

$$\mathrm{Ball}(x', r') \subset \mathrm{Ball}(x, r).$$

In English, if x' is a point contained in an open ball (centered at a possibly different point x), then one can always find an open ball *centered at* x' contains in the original open ball.

Proof. We set

$$r' = r - d_X(x, x').$$

Indeed, if any other point w is contained in $\mathrm{Ball}(x', r')$, the triangle inequality says

$$d_X(x, w) \leq d_X(x, x') + d_X(x', w)$$

but the righthand side satisfies

$$d_X(x, x') + d_X(x', w) < d_X(x, x') + r' = d_X(x, x') + r - d_X(x, x') = r.$$

So we are finished. $\square$

4.4.2 Proof of Theorem 4.4.1.

Recall that to prove two statements are equivalent, we need to prove that one implies the other, and vice versa.

Proof. We first prove that (1) implies (2). Let $V \subset Y$ be open. We must prove that if f is continuous, then $f^{-1}(V)$ is open. So choose $x \in f^{-1}(V)$. The goal is to find some ball B_x of positive radius containing x and contained in $f^{-1}(V)$. (If we can do this for all $x \in f^{-1}(V)$, then we have that

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} B_x$$

and this would show that $f^{-1}(V)$ is open.)

Since $V \subset Y$ is open, it can be written as a union of open balls. In particular, there is some open ball $\mathrm{Ball}_Y(y, r)$ such that

$$f(x) \in \mathrm{Ball}_Y(y, r) \subset V.$$

Claim One: There exists some $\epsilon > 0$ such that $\text{Ball}_Y(f(x), \epsilon) \subset \text{Ball}_Y(y, r)$. Indeed, this is the reason I introduced Lemma 4.4.2. Using that lemma, Claim One is proven.

Now, by the continuity of f, there exists δ such that $d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon$; in particular, there is some δ such that the ball of radius δ centered at x is contained in $f^{-1}(\text{Ball}_Y(f(x), \epsilon))$. But we know that

$$\text{Ball}_Y(f(x), \epsilon) \subset \text{Ball}_Y(y, r) \subset V$$

so in particular,

$$\text{Ball}_X(x, \delta) \subset f^{-1}(V).$$

We have accomplished our goal. This proves that (1) implies (2).

Now let us prove that (2) implies (1). If we assume that the preimage of any open set is open, we must prove that f is continuous.

So fix $x \in X$ and fix $\epsilon > 0$. Then the open ball $V = \text{Ball}_Y(f(x), \epsilon)$ is an open subset of Y, so in particular, we know that $f^{-1}(V)$ is an open subset of X. By definition, then, it can be written as the union of open balls—in particular, there is some open ball containing x. By Lemma 4.4.2, we may choose this ball to be centered at x, and we will write its radius as δ. By construction, this ball is contained in $f^{-1}(V)$, and we thus have

$$f(\text{Ball}_X(x, \delta)) \subset V = \text{Ball}_Y(f(x), \epsilon).$$

In other words, if any point x' is within δ of x, it follows that $d_Y(f(x), f(x')) < \epsilon$. This completes the proof. $\square$

Lecture 5

Tuesday, September 10th

Today, you will do some in-class exercises.

Work individually, or in groups, as you like.

These exercises will prepare you for a quiz you will have at the end of today's class.

5.1 Practice with metrics

Let $X = \mathbb{R}^n$. An element $x \in X$ is thus represented by a sequence of n real numbers, and we will denote this sequence by

$$x = (x_1, x_2, \dots, x_n)$$

where x_i denotes the ith coordinate of x.

Recall we have defined the following:

1. $d_{std}(x, x') = \sqrt{\sum_{i=1}^{n} (x'_i - x_i)^2}$
2. $d_{discrete}(x, x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x'. \end{cases}$
3. $d_{taxi}(x, x') = \sum_{i=1}^{n} |x'_i - x_i|$
4. $d_{l^\infty}(x, x') = \max_{i=1}^{n} |x'_i - x_i|$

Example 5.1.1. Let $n = 3$ and let

$$x = (1, \pi, 8), \qquad x' = (3, \pi + 5, 16).$$

The we have that

$$\begin{aligned}
d_{std}(x, x') &= \sqrt{\sum_{i=1}^{3} (x'_i - x_i)^2} \\
&= \sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + (x'_3 - x_3)^2} \\
&= \sqrt{(3 - 1)^2 + ((\pi + 5) - \pi)^2 + (16 - 8)^2} \\
&= \sqrt{2^2 + 5^2 + 8^2} \\
&= \sqrt{4 + 25 + 64} \\
&= \sqrt{93}
\end{aligned}$$

$$d_{discrete}(x, x') = 1$$

$$\begin{aligned}
d_{taxi}(x, x') &= \sum_{i=1}^{3} |x'_i - x_i| \\
&= |x'_1 - x_1| + |x'_2 - x_2| + |x'_3 - x_3| \\
&= |(3 - 1)| + |(\pi + 5) - \pi| + |16 - 8| \\
&= 2 + 5 + 8 \\
&= 15
\end{aligned}$$

$$\begin{aligned}
d_{l^\infty}(x, x') &= \max_{i=1}^{n} |x'_i - x_i| \\
&= \max\{|x'_1 - x_1|, |x'_2 - x_2|, |x'_3 - x_3|\} \\
&= \max\{|(3 - 1)|, |(\pi + 5) - \pi|, |16 - 8|\} \\
&= \max\{2, 5, 8\} \\
&= 8
\end{aligned}$$

5.1.1

Let $n = 4$ and let

$$x = (1, 3, 7, 16), \qquad x' = (3, 8, 2, 1).$$

Compute each of the following:

1. $d_{std}(x, x')$
2. $d_{discrete}(x, x')$
3. $d_{taxi}(x, x')$
4. $d_{l^\infty}(x, x')$.

5.1.2

Let (X, d) be a metric space. Fix $x \in X$ and $r > 0$. Recall that $\mathrm{Ball}_d(x, r)$ is the set of all points x' satisfying $d(x, x') < r$.

Let $X = \mathbb{R}^2$ and let $x = (1, 3)$. Draw each of the following:

1. $\mathrm{Ball}_{d_{std}}(x, 3)$
2. $\mathrm{Ball}_{d_{discrete}}(x, 3)$
3. $\mathrm{Ball}_{d_{taxi}}(x, 3)$
4. $\mathrm{Ball}_{d_{l^\infty}}(x, 3)$

Make sure you can justify what you draw.

5.1.3

This is a challenge problem.

Let $X = \mathbb{R}^n$, and for every real number $p \geq 0$, let us define a function

$$d_{l^p}(x, x') = \left(\sum_{i=1}^{n} (|x_i' - x_i|)^p \right)^{1/p}.$$

Question: For which values of p does this define a metric on X?

(Note that if $p = 1$, this is the taxicab metric. If $p = 2$, it is the standard metric.)

You can draw picture to try an make a guess, but this is a hard problem. So good luck and have fun!

5.2 Quiz

5.2.1 (12 points)

Let $n = 4$ and let

$$x = (2, 3, 1, 4), \qquad x' = (3, 8, 2, 1).$$

Compute each of the following:

(a) $d_{std}(x, x')$

(b) $d_{discrete}(x, x')$

(c) $d_{taxi}(x, x')$

(d) $d_{l^\infty}(x, x')$.

In case you need it:

- $d_{std}(x, x') = \sqrt{\sum_{i=1}^n (x_i' - x_i)^2}$
- $d_{discrete}(x, x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x'. \end{cases}$
- $d_{taxi}(x, x') = \sum_{i=1}^n |x_i' - x_i|$
- $d_{l^\infty}(x, x') = \max_{i=1}^n |x_i' - x_i|$

5.2.2 Extra Credit. 10 points

Let (X, d) be a metric space. Which of the following is true?

(a) The function
$$d_{\log}(x, x') = \log |d(x, x') + 1|$$
defines a metric on X.

(b) The function
$$d_{\exp}(x, x') = \exp(d(x, x')) - 1$$
defines a metric on X.

You will get a full 5 points for every complete proof, or counterexample. You will not get any negative points for an incorrect response.

Lecture 6

Thursday, September 12th

Today, you are going to explore how continuity plays with convergence of sequences. You had a warm-up to this in your first homework assignment, where you explored convergence for sequences in the real line.

Definition 6.0.1. Let (X, d) be a metric space. Fix a sequence

$$x_1, x_2, x_3, \ldots$$

of points in X. We say that the sequence *converges* to $x \in X$ if the following holds:

For every $\epsilon > 0$, there exists a number N so that

$$i > N \implies d(x_i, x) < \epsilon.$$

Remark 6.0.2. In plain English, this means that if you want to be ϵ-close to x, you just need to be at least N-far-along in the sequence.

Definition 6.0.3. We say a sequence is *convergent* if there exists an element x to which it converges.

Feel free to work individually, or in groups. If you get stuck, make sure to figure out where, how, or why you get stuck.

6.1

Suppose that a sequence $x_1, x_2, \ldots$ converges to x, and also converges to x'. Show that $x = x'$.

That is, show that if a sequence converges, it converges to a unique point.

(What properties of metric space did you use in your proof?)

(Make sure that in your proof, you are never subtracting or adding elements of X, but you are only subtracting/adding elements of $\mathbb{R}$.)

6.2

Let (X, d_X) and (Y, d_Y) be metric spaces, and fix a function $f : X \to Y$. Show that the following are equivalent:

(a) f is continuous.

(b) If $x_1, \ldots,$ is a convergent sequence, then so is $f(x_1), \ldots$.

(c) If $x_1, \ldots,$ is a sequence converging to x, then $f(x_1), \ldots$ is a sequence converging to $f(x)$.

6.3 (Optional)

Can you think of two metric spaces (X, d_X) and (Y, d_Y), and a continuous bijection $f : X \to Y$, such that the inverse function $f^{-1} : Y \to X$ is not continuous?

Can you think of such a thing when $X = Y = \mathbb{R}^n$ and d_X, d_Y are one of the non-discrete metrics we've discussed? (I.e., when we choose d_X and/or d_Y to be one of $d_{std}, d_{taxi}, d_{l^\infty}$?) Why or why not?

Lecture 7

Tuesday, September 17th

7.1 Non-negativity of metrics

Exercise 7.1.1. Let (X, d) be a metric space. Show that $d(x, x') \geq 0$ for any $x, x' \in X$.

Proof. Use the triangle inequality for $x = x' = x''$. Then

$$0 = d(x, x'') \leq d(x, x') + d(x', x'') = 2d(x, x').$$

So (dividing the beginning and the end by 2), we see $d(x, x') \geq 0$. □

7.2 Simplifying the verification of continuity

Exercise 7.2.1. Let (X, d_X) and (Y, d_Y) be metric spaces, and fix a function $f : X \to Y$. Show the following are equivalent:

1. For any open set V, $f^{-1}(V)$ is an open set.

2. For any open ball $\mathrm{Ball}_\epsilon(y)$, $f^{-1}(V)$ is an open set.

Proof. 1 implies 2: Any open ball is an open set; so setting $V = \mathrm{Ball}_\epsilon(y)$, 1 implies 2.

2 implies 1: We know that any open set V is a union of open balls, so

$$V = \bigcup \mathrm{Ball}_\epsilon(y)$$

for some collection of open balls. Thus

$$f^{-1}(V) = \bigcup f^{-1}(\mathrm{Ball}_\epsilon(y))$$

where the righthand side is a union of open subsets of X. In homework you proved that any union of open subsets is again open. Thus $f^{-1}(V)$ is open. This proves 2 implies 1. $\square$

Putting together everything, we have proven the following so far in this class:

Theorem 7.2.2. Let (X, d_X) and (Y, d_Y) be metric spaces. Fix a function $f : X \to Y$. The following are equivalent:

1. f is continuous.
2. The preimage of any open subset of Y is an open subset of X.
3. The preimage of any open ball of Y is an open subset of X.
4. f sends convergent sequences in X to convergent sequences in X.

7.3 From homework

Let X and Y be metric spaces. Define the following metric on $X \times Y$:

$$d((x,y),(x',y')) = d_X(x,x') + d_Y(y,y').$$

Show that the projection map $(x, y) \mapsto x$ is continuous.

Proof. There are two ways you could do this.

(i) Using ϵ-δ. Fix an element of the domain, (x_0, y_0), and some $\epsilon > 0$. We must show the existence of some δ such that

$$d((x_0,y_0),(x,y)) < \delta \implies d_X(x_0,x) < \epsilon. \tag{7.3.0.1}$$

I claim any $\delta \leq \epsilon$ works. This is because

$$d((x_0,y_0),(x,y)) = d_X(x_0,x) + d_Y(y_0,y) \geq d_X(x_0,x).$$

Where the last inequality follows because $d_Y(y_0, y) \geq 0$. Thus (7.3.0.1) follows if $\delta \leq \epsilon$.

(ii) Using that a function is continuous if and only if the preimage of any open set is open.

Let's call our function π, so $\pi(x, y) = x$. By the exercise earlier this lecture, we must verify that for any open ball in X, the preimage is an open subset of X.

So fix an open ball $B_\epsilon(x) \subset X$. Here, $x \in X$ and $\epsilon > 0$. By definition of π, the preimage of this is the set of all pairs (x, y) such that x is in the ball, and y is arbitrary. This can be written as a union of open balls as well:

$$\bigcup_{r, x' \text{ such that } B_r(x') \subset B_\epsilon(x)} \bigcup_{y \in Y} B_r((x', y))$$

$\square$

Remark 7.3.1. Note that (ii) seems a little bit more complicated. Regardless, we have two very different-looking proofs of the same fact. This is a good sign that the equivalent criteria for continuity are appreciably different, and hence useful! (Having two very different ways to tackle the same problem is a gift.)

7.4 Intuition for open sets in metric spaces

What is the intuition for how to think about an open set in a metric space? Recall that an open set in a metric space is any subset that can be written as a union of open balls. Recall also that we proved the following lemma last time I lectured: If x is contained in some open ball $\text{Ball}_{\epsilon'}(x')$, then there is another open ball $\text{Ball}_\epsilon(x)$, centered at x, taht is contained in $\text{Ball}_{\epsilon'}(x')$.

Corollary 7.4.1 (Of the Lemma)**.** Let (X, d) be a metric space and let $U \subset X$ be an open subset. Then for any $x \in U$, there exists an open ball centered at x.

In fact, we have

Proposition 7.4.2. Let (X, d_X) be a metric space and fix a subset $U \subset X$. The following are equivalent:

1. U is an open subset.

2. For any $x \in U$, U contains an open ball of some (small) positive radius centered at x. That is, there exists $\delta > 0$ so that $\text{Ball}(x; \delta) \subset U$.

Proof. 1 implies 2. We use the (re)centering lemma from last time I lectured. If U is open, it's a union of open balls:

$$U = \bigcup_{\alpha} \mathrm{Ball}(x_\alpha, \delta_\alpha)$$

where α indexes some collection of centers x_α and radii δ_α. Thus for any $x \in U$, there is some α so that $x \in \mathrm{Ball}(x_\alpha, r_\alpha)$. By the (re)centering lemma, this means that there is some δ so that

$$\mathrm{Ball}(x, \delta) \subset \mathrm{Ball}(x_\alpha, r_\alpha).$$

In particular,

$$\mathrm{Ball}(x, \delta) \subset U.$$

2 implies 1. For every $x \in U$, choose δ_x so that $\mathrm{Ball}(x, \delta_x) \subset U$. Then we have that

$$\bigcup_{x \in U} \mathrm{Ball}(x, \delta_x) = U.$$

To see this equality, note that the righthand side is contained in U (because a union of subsets is still a subset). The lefthand side is contained in the righthand side: Given $x' \in U$, note that $\mathrm{Ball}(x', \delta_{x'})$ is one of the balls in the union on the lefthand side, and in particular, $x' \in \mathrm{Ball}(x', \delta_{x'})$. □

Remark 7.4.3. This proposition is supposed to give you intuition for what open sets look like: U is open if and only if for any $x \in U$, x has "enough wiggle room," or "enough breathing room" in U. By "wiggle room," I mean there is some δ so that x can move around in an open ball of radius δ without leaving U.

Warning 7.4.4. The notion of being an open subset depends on the metric space we are in. That is, when we say "U is open," we have a metric space in mind of which U is a subset.

Example 7.4.5 (Of sets that are not open)**.** Let $A \subset \mathbb{R}$ be a closed bounded interval. Then A is not open. For example, at the endpoint x of A, no open interval about x is fully contained in A. (To see this: Any open interval $(x - \epsilon, x + \epsilon)$ contains some element larger than, or some element less than, x. But because x is an endpoint, it is either the minimal or maximal element of A. Without loss of generality, assume x is minimal. Then A could not contain an element less than x itself.)

Likewise, let $A \subset \mathbb{R}^2$ be a closed bounded interval. Then A is not open. For example, even if x is in the interior of A, no open ball of $\mathbb{R}^2$ fits inside A.

7.5 Beyond metric spaces

So just as we're getting used to metric spaces, I want to suggest to you that the zoo of metric spaces is too constricting. For the next half an hour, I'd like you to think about the following problems:

1. Can you give the circle a metric space structure? How about the sphere? How? Can you give any subset of $\mathbb{R}^n$ a metric space structure? Are they meaningful?

2. Consider the "set of all lines through the origin" in $\mathbb{R}^2$. Make sure you think about what this means. Can you give this a metric space structure?

3. Consider the shape you would get if you were to take a sheet of paper, and carefully glue/tape two opposing edges together. Can you give this a metric space structure?

What we saw in class is that the first example is not so bad to tackle: Any subset $A \subset X$ of metric space (X, d_X) can be given a metric space structure.

Definition 7.5.1. Let (X, d_X) be a metric space and let $A \subset X$ be a subset. The *subset metric*, or *induced metric* on A is

$$d_A(x, x') := d_X(x, x').$$

The subset metric is indeed a metric on A. I'll the proof to you as an exercise.

In class we had some difficulty with the set of lines in $\mathbb{R}^2$. We had the insight to try to assign to each line an "angle," but this assignment didn't seem continuous. And depending on how we made the shape obtained by gluing a sheet of paper along its edges, the metrics could be different.

But the notion of having "wiggle room"—we were supposed to discover—is one we can articulate better.

Main idea: Sometimes, it's easier to think about wiggle room (open sets) than it is to think about metrics.

We'll begin next time with topological spaces.

Lecture 8

Thursday, September 19th

8.1 Some announcements

8.1.1 Collaboration policy

Some of your homeworks are far too similar. Please read the collaboration policy I've put online. In short, you can collaborate, and you can consult sources, but when you are *writing* (whether on a laptop, phone, or paper) your homework for submission, you must be alone and not using any resources.

8.1.2 Multiple choice

From now on, multiple choice responses for homework will be submitted online. Links will be on the website every week. Don't let the convenient format fool you—the multiple choices are often the hardest part of the homework. You do not need to hand in paper submissions. These are always do before 1:50 PM on Tuesdays. Anything submitted after 1:50 PM will not be accepted.

8.1.3 Next homework

For the next proof homework, I will scan copies of your submissions and share them with the class. You will get to see the work of other classmates; and your classmates will see your submissions, too.

Put your names on the homeworks; I will anonymize them the best I can when I share with class.

8.2 More on open sets

Last time we saw the following intuition for open sets in metric spaces: A subset $U \subset X$ is open if and only if for every $x \in U$, there is a (small) open ball of positive radius, centered at x and contained in U. We interpreted this to mean that a set U is open if and only if every $x \in U$ has "wiggle room" inside U.

Remark 8.2.1. In class discussions and in homeworks, I have also seen some of you engage with the notion of a "boundary" of a set. We will talk about this in due time.

A philosophy I've mentioned more than once: To study objects, we need to study the functions between them. This philosophy is not at all obvious in your earliest serious math classes, but you've at least seen that there are many interesting functions $f : \mathbb{R} \to \mathbb{R}$ to explore (in calculus class, for example). But it is an important philosophy regardless.

What we have seen so far in class—though you may not have noticed it—is that for a map $(X, d_X) \to (Y, d_Y)$ to be continuous certainly depends on the metrics in play, but that it doesn't depend on the entire data of the metrics. For example, we have seen that to check whether a function is continuous, we only need to check whether preimages of open sets are open.

In other words: *Continuity depends only on open sets.*

Remark 8.2.2. Combining our intuition of "wiggle room" with our "open set" test for continuity, we arrive at the following intuition. A function $f : X \to Y$ is continuous if and only if: Every $x \in X$ has wiggle room[1] to stay within any specified wiggle room[2] of $f(x)$.

So here's a natural question: Does the collection of open sets of a metric space "remember" the metric of the metric space? Put another way, if (X, d_X) is a metric space, and $\mathcal{T}$ is its collection of open sets, does $\mathcal{T}$ determine d_X? The answer is no:

Theorem 8.2.3. Let $\mathcal{T}_{std}$ denote the collection of open sets in $\mathbb{R}^n$ for d_{std}.

Likewise, we let $\mathcal{T}_{taxi}$ and $\mathcal{T}_{l^\infty}$ denote the collections of open sets in $\mathbb{R}^n$ with respect to d_{taxi} and d_{l^∞}.

[1] i.e., a ball of radius δ

[2] a ball of radius ϵ

Then
$$\mathcal{T}_{std} = \mathcal{T}_{taxi} = \mathcal{T}_{l^\infty}.$$

Proof. I will sketch a proof.

Let $U \subset \mathbb{R}^n$ be open with respect to the standard metric. Then for any $x \in U$, there is some δ_{std} so that
$$\mathrm{Ball}_{std}(x; \delta_{std}) \subset U.$$
But given δ_{std}, I claim there exists δ_{taxi} such that
$$\mathrm{Ball}_{taxi}(x; \delta_{taxi}) \subset \mathrm{Ball}_{std}(x; \delta_{std}).$$
That is, any open ball in the standard metric (centered at x) contains an open ball in the taxicab metric (also centered at x). I will just draw a picture of this in class. (It turns out you could take $\delta_{taxi} = \delta_{std}$ because the open ball in the standard metric is convex.)

We conclude that if U is open with respect to d_{std}, it is open with respect to d_{taxi}.

Conversely, let U be open with respect to d_{taxi}. Then for any $x \in U$, there is an open ball $\mathrm{Ball}_{taxi}(x, \delta_{taxi})$ centered at x and contained in U. You can check that the shortest standard distance from x to a "wall" of this taxicab-ball (which is a diamond-shaped region) is given by
$$\delta_{taxi}\sqrt{1/n}$$
where n is the dimension of $\mathbb{R}^n$. Thus we see that
$$\mathrm{Ball}_{std}(x, \delta_{taxi}\sqrt{1/n}) \subset \mathrm{Ball}_{taxi}(x, \delta_{taxi}) \subset U$$
so U is open with respect to d_{std} as well.

This shows $\mathcal{T}_{std} = \mathcal{T}_{taxi}$.

A similar proof shows that $\mathcal{T}_{std} = \mathcal{T}_{l^\infty}$. $\square$

Thus, although the metrics on $\mathbb{R}^n$ are distinct, they give rise to the *same* collection of open sets.

This proves the following:

Corollary 8.2.4. The identity functions
$$\begin{array}{cc} (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{taxi}) & (\mathbb{R}^n, d_{std}) \to (\mathbb{R}^n, d_{l^\infty}) \\ (\mathbb{R}^n, d_{taxi}) \to (\mathbb{R}^n, d_{std}) & (\mathbb{R}^n, d_{taxi}) \to (\mathbb{R}^n, d_{l^\infty}) \\ (\mathbb{R}^n, d_{l^\infty}) \to (\mathbb{R}^n, d_{taxi}) & (\mathbb{R}^n, d_{l^\infty}) \to (\mathbb{R}^n, d_{std}) \end{array}$$
are all continuous.

Proof. The preimage of U is given by U. Moreover, if U is open with respect to one of the metrics above, it is also open with respect to any of the others by the previous result. This shows that the preimage of any open subset is open, hence the identity function is continuous. □

Corollary 8.2.5. Let (Y, d_Y) be a metric space. Let $f : \mathbb{R}^n \to Y$ be a function. Then f is continuous with respect to the standard (or taxi, or l^∞) metric if and only if it is continuous with respect to any of three metrics above (standard, taxi, or l^∞).

Proof. Let $V \subset Y$ be open. Then $f^{-1}(V)$ is open with respect to one of the three metrics above if and only if it is open with respect to all of them. □

Remark 8.2.6. This is the first hint that a notion of distance *helps* detect continuity, but continuity does not depend on a notion of distance! How great is that?

8.3 Constructing new spaces

So we have seen two kinds of things with the word "space" in the name. Let me recall them both:

Definition 8.3.1. A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function satisfying:

(0) For all $x, x' \in X$, $d(x, x') = 0 \iff x = x'$.

(1) For all $x, x' \in X$,$d(x, x') = d(x', x)$, and

(2) For all $x, x', x'' \in X$, we have that

$$d(x, x') + d(x', x'') \geq d(x, x'').$$

Definition 8.3.2. A *topological space* is a pair

$$(X, \mathcal{T})$$

where X is a set, and $\mathcal{T}$ is a collection of subsets of X, satisfying the following three properties:

1. Both $\emptyset$ and X are elements of $\mathcal{T}$.

2. If $U_1, \ldots, U_k$ is a finite collection of elements of $\mathcal{T}$, then the intersection $U_1 \cap \ldots \cap U_k$ is an element of $\mathcal{T}$. That is, $\mathcal{T}$ is closed under finite intersections.

3. If $\mathcal{A}$ is an arbitrary set and $\mathcal{A} \to \mathcal{T}$ is a function (so for every $\alpha \in \mathcal{A}$ we have an element $U_\alpha \in \mathcal{T}$) then the union
$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$
is also in $\mathcal{T}$. That is, $\mathcal{T}$ is closed under arbitrary unions.

Definition 8.3.3. We will call $\mathcal{T}$ a *topology* on X, and any element $U \in \mathcal{T}$ will be called an *open* subset of X.

Example 8.3.4. Let (X, d_X) be a metric space, and let $\mathcal{T}$ be the collection of open sets determined by d_X. (That is, $U \in \mathcal{T}$ if and only if U is a union of open balls.) Then you proved in homework that $(X, \mathcal{T})$ is a topological space.

Definition 8.3.5. Let (X, d_X) be a metric space and let $\mathcal{T}$ be the collection of open sets with respect to d_X. We say that $\mathcal{T}$ is the topology *induced* by the metric.

Remark 8.3.6. Note that these definitions have incredibly different flavors. For example, the notion of metric space depends very much on *numerical* or *quantitative* statements—meaning we rely on properties of the real line. (For example, we rely on the fact that we know how to add elements of $\mathbb{R}$, and on the fact that we knowhow to compare the sizes of elements of $\mathbb{R}$.)

In contrast, the definition of topological space is much more barren—it does not even need mention of the real line. It only relies on the fact that we can consider subsets of a set X, and that we can take unions and intersections of subsets.

Remark 8.3.7. This barrenness is both a strength and downside of the definition. The downside is that it takes a lot to get used to. But the strength is that one can speak of many interesting phenomena under the same umbrella—even if we cannot measure distances. In some sense, it frees us from our dependence on distance.

Though I have not framed things this way, we have seen that we can construct new metric spaces from old ones. For example:

1. If (X, d) is a metric space, then any subset $A \subset X$ can be made into a metric space under the subset metric.

2. If (X, d_X) and (Y, d_Y) are metric spaces, then the product $X \times Y$ can be made into a metric space.

Warning 8.3.8. There are many non-equivalent ways to make $X \times Y$ into a metric space. This was explored a little bit in one of the exercises in-class; and it already visible in the case of $X = Y = \mathbb{R}$.

For example, $\mathbb{R}^2$ has many different metrics, as we've seen. In parallel, $X \times Y$ can be given any of the following metrics:

1. $d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$. (If $X = Y = (\mathbb{R}, d_{std})$, this gives rise to the taxicab metric in $\mathbb{R}^2$.)

2. $d((x, y), (x', y')) = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}$. (If $X = Y = (\mathbb{R}, d_{std})$, this gives rise to the standard metric in $\mathbb{R}^2$.)

3. $d((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}$. (If $X = Y = (\mathbb{R}, d_{std})$, this gives rise to the l^∞ metric in $\mathbb{R}^2$.)

Remark 8.3.9 (Quotients will become easy)**.** Last time, we saw an example where given $S^1 \subset \mathbb{R}^2$, it was very easy to construct a metric on S^1 by just using the subset metric. But when we had to think about the set of lines in $\mathbb{R}^2$, or the cylindrical shape formed by gluing two edges of a sheet of paper together, it was not so obvious how to define a metric that everybody agreed on.

These latter two examples are examples of "quotient spaces." It turns out that while it is very difficult to naturally put metrics on quotient spaces, it is very easy to put a topology on them.

For now, let's see that it is easy to construct topologies on subsets and on product sets, just as it was easy to construct metrics on them. If you believe that quotients are also easy places to construct topologies, we see that working with topological spaces has a lot of pros:

1. It's easy to construct new spaces, and

2. The notion of continuity can be expressed purely in terms of open sets.

8.3.1 Subset topology

Exercise 8.3.10. Let $(X, \mathcal{T})$ be a topological space, and fix a subset $A \subset X$. Define

$$\mathcal{T}_A$$

to consist of those subsets $W \subset A$ such that $W = U \cap A$ for some $U \in \mathcal{T}$. (That is, a subset of A is declared open if and only if it is the intersection of A with an open set of X.)

Prove that $\mathcal{T}_A$ is a topology on A.

Proof. We must verify the three properties:

1. $\emptyset \in \mathcal{T}_A$ because $\emptyset \cap A = \emptyset$ and $\emptyset \in \mathcal{T}$. Likewise, $A \in \mathcal{T}_A$ because $A = X \cap A$ and $X \in \mathcal{T}$.

2. Consider a finite collection $W_1, \ldots, W_k \in \mathcal{T}_A$. For each W_i, we know
$$W_i = U_i \cap A$$
for some $U_i \in \mathcal{T}$. Then
$$W_1 \cap \ldots \cap W_k = (U_1 \cap A) \cap \ldots (U_k \cap A) = (U_1 \cap \ldots \cap U_k) \cap A$$
and this last term is an intersection of an open set $U_1 \cap \ldots \cap U_k$ with A. (Note that the intersection of the U_i is open because $\mathcal{T}$ is a topology.)

3. Now fix an arbitrary collection $\{W_\alpha\}_{\alpha \in \mathcal{A}}$. Then for each α, there exists some $U_\alpha \in \mathcal{T}$ such that $W_\alpha = U_\alpha \cap A$. So
$$\bigcup_\alpha W_\alpha = \bigcup_\alpha (U_\alpha \cap A) = (\bigcup_\alpha U_\alpha) \cap A.$$
The set in the parentheses is open because $\mathcal{T}$ is a topology; hence this intersection is in $\mathcal{T}_A$ by definition of $\mathcal{T}_A$.

$\square$

Definition 8.3.11. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ a subset. The topology $\mathcal{T}_A$ on A is called the *subset topology* on A.

Remark 8.3.12. The above proof is typical of the kinds of proofs you'll see in general topology—the formulas are formulas involving intersections and unions of sets, and the way we index these intersections and unions take a bit of getting used to. This is inherent in the definition of topological space: Because the definition only uses tools of sets (intersections, unions, et cetera) so too will the proofs use such tools.

This is contrast to metric spaces, where we got to use real numbers, additions, and inequalities.

Example 8.3.13. Let $X = \mathbb{R}^2$ with the standard topology (induced by the standard metric—or the taxicab metric, or the l^∞ metric). And let $A = S^1 \subset X$ be the unit circle. Let us endow A with the subset topology.

Then a subset $W \subset A$ is open if and only if it is the intersection of an open set of $\mathbb{R}^2$ with the circle. You should try drawing some examples. For instance, any open interval on the circle is an open subset. The circle itself is an open subset, too.

Note that it is impossible for you to draw *every* open subset of $\mathbb{R}^2$; there are just too many. One of the powers of the definition of topological space is that you don't need to know what all open subsets are. Often, you'll only need to know some basic open subsets that all other open subsets are made of; this will lead us to the notion of a basis for a topology, and we'll see that in a week or two.

Remark 8.3.14. Suppose that (X, d_X) is a metric space; then we know that a subset $A \subset X$ inherits a metric space structure. Since (A, d_A) is a metric space, one can induce a topology from the metric.

On the other hand, we have just seen that A can inherit a topology from X (without passing through a metric on A).

It turns out that these two topologies are the same. I'll leave this as an exercise to you.

8.3.2 Product topology

Definition 8.3.15. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let us define

$$\mathcal{T}$$

to be the collection of subsets of $X \times Y$ that can be expressed as unions of sets of the form $U \times V$, where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$.

We call this the *product topology* on $X \times Y$.

Exercise 8.3.16. Show that $\mathcal{T}$ is a topology on $X \times Y$.

Proof. 1. The empty set can be written as $\emptyset \times \emptyset$, so the empty set is in $\mathcal{T}$. Like wise, $X \times Y$ is in $\mathcal{T}$ because X and Y are open sets of X and Y, respectively.

2. We first note that if U, U' and V, V' are open subsets of X and Y, respectively, then
$$(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V').$$
To see this, note that (x, y) is in the intersection if and only if $x \in U \cap U'$ and $y \in V \cap V'$.

So suppose that $W \in \mathcal{T}$, so
$$W = \bigcup_{\alpha \in \mathcal{A}} U_\alpha \times V_\alpha$$
is some union of products of open subsets of X and Y. Fix another open subset
$$W' = \bigcup_{\beta \in \mathcal{B}} U_\beta \times V_\beta.$$
Then[3]
$$W \cap W' = \bigcup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} (U_\alpha \times V_\alpha) \cap (U_\beta \times V_\beta) = \bigcup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} (U_\alpha \cap U_\beta) \times (V_\alpha \cap V_\beta)$$
is a union of products of open sets.

3. If $\{W_\alpha\}_{\alpha \in \mathcal{A}}$ is some collection of open sets, each W_α can be expressed as a union
$$W_\alpha = \bigcup_{\gamma \in \mathcal{C}_\alpha} U_\gamma \times V_\gamma.$$
Hence
$$\bigcup_{\alpha \in \mathcal{A}} W_\alpha = \bigcup_{\alpha \in \mathcal{A}, \gamma \in \mathcal{C}_\alpha} U_\gamma \times V_\gamma$$
is a union of products of open sets as well. This shows $\bigcup_\alpha W_\alpha \in \mathcal{T}$.

□

[3]This is a careful application of facts about sets. Note that for something to be in the intersection of W and W', it must be contained in some $U_\alpha \times V_\alpha$ and some $U_\beta \times V_\beta$. Likewise, if an element in the intersection of some $U_\alpha \times V_\alpha$ and some $U_\beta \times V_\beta$, it is in $W \cap W'$. In other words, if I take the intersection of $U_\alpha \times V_\alpha$ with $U_\beta \times V_\beta$ for every α, β, and consider the union of these intersections, I recover $W \cap W'$.

Lecture 9

Tuesday, September 24th

9.1 Intro to quotient spaces

In homework, you showed the following:

Fix a topological space $(X, \mathcal{T}_X)$ and a surjection $p : X \to Y$. Then you can give Y a topology. Moreover, this topology satisfies the following property:

If $(Z, \mathcal{T}_Z)$ is another topological space, then a function $f : Y \to Z$ is continuous if and only if the composition $p \circ f : X \to Z$ is continuous.

Remark 9.1.1. The power of this statement is that you can check the continuity of f by *checking something between X and Z*, not between Y and Z. If you have more information about X then about $(Y, \mathcal{T}_Y)$, this is a very useful technique.

But where do surjections $X \to Y$ come from? There is a natural source: When Y is a quotient of X. Today's goal is to explain this.

In the last couple classes, we've tried to consider the two following sets:

1. The set of all lines through the origin (in $\mathbb{R}^2$), and

2. The cylinder-like gadget one gets by gluing two edges of a sheet of paper together.

I claim that we can realize both sets as the Y in the quotient space construction, and thereby endow these sets with topologies.

Example 9.1.2. In the example of the cylinder-like gadget, the sheet of paper surjects onto the cylinder like object. So the sheet of paper is the X and the cylinder-like gadget is the Y.

It is not at obvious what X one can take to surject onto the set of lines in $\mathbb{R}^2$ through the origin.

9.2 Equivalence relations and quotient sets

I want to tell you how to take a set X and "glue" some of its elements together.

Remark 9.2.1. This is imprecise, but is meant to give intuition. In what follows, the following expressions will roughly mean the same thing:

1. To "glue" two points of X together.
2. To make two points of X equal.
3. Identifying two points of X.

The mathematical toolkit we have for identifying points of X is called an equivalence relation.

Definition 9.2.2. Let X be a set. An *equivalence relation* on X is a choice of subset

$$E \subset X \times X$$

satisfying the following:

(0) (Reflexivity.) For every $x \in X$, the element (x, x) must be in E.

(1) (Symmetry.) For every $x, x' \in X$, if $(x, x') \in E$, then (x', x) is in E.

(2) (Transitivity.) For every $x, x', x'' \in X$, if $(x, x') \in E$ and $(x', x'') \in E$, then $(x, x'') \in E$.

Notation 9.2.3. Let $E \subset X \times X$ be an equivalence relation on X. Then we will write

$$x \sim x'$$

and say "x is related to x'" whenever $(x, x') \in E$.

Example 9.2.4. In the $\sim$ notation, the above three properties of an equivalence relation may be written as

(0) (Reflexivity.) For every $x \in X$, $x \sim x$.

(1) (Symmetry.) For every $x, x' \in X$, $x \sim x' \implies x' \sim x$.

(2) (Transitivity.) For every $x, x', x'' \in X$, $x \sim x'$ and $x' \sim x''$ implies $x \sim x''$.

Example 9.2.5. The prototypical example of an equivalence relation is the *equality* relation. That is,

$$x \sim x' \iff x = x'.$$

In this example, E is equal to the set of all pairs (x, x). (That is, those (x, x') such that $x = x'$.) In terms of the intuition that an equivalence relation tells you which elements to identify, this relation tells you to introduce no new identifications—i.e., you only glue a point to itself, so you are not gluing any non-distinct points together.

Example 9.2.6. Another example of an equivalence relation is to glue everything together—i.e., to glue any two points to each other. That is,

$$x \sim x' \text{ for any } x, x' \in X.$$

That is, E is equal to $X \times X$ itself.

Now let us give a name for the set of all points that are identified to each other.

Definition 9.2.7. Fix a set X and an equivalence relation $E \subset X \times X$. An *equivalence class* of E is a subset $A \subset X$ satisfying the following:

(0) A is non-empty.

(1) If $x \in A$ and $x \sim x'$, then $x' \in A$.

(2) If $x, x' \in A$, then $x \sim x'$.

Exercise 9.2.8. Fix an equivalence relation E on X and let $A_1, A_2 \subset X$ be two equivalence classes. Show that if there exists an element $x \in A_1 \cap A_2$, then $A_1 = A_2$.

Proof. Let $[x]$ be the collection of those $x' \in X$ such that $x \sim x'$. I first claim that if any equivalence class A contains x, then $A = [x]$.

$A \subset [x]$ follows from property (2) of an equivalence class (Definition 9.2.7).

$[x] \subset A$ follows from property (1) of an equivalence class.

Thus $A_1 = [x] = A_2$ and we are finished. □

What the above exercise tells us is that any equivalence relation on X *partitions* X. That is, it allows us to write X as a union of subsets called equivalence classes:

$$X = \bigcup A$$

Moreover, if $A \neq A'$, then $A \cap A' = \emptyset$. Thus X is a union of subsets that are *disjoint* from one another.

Definition 9.2.9. Let X be a set and $E \subset X \times X$ an equivalence relation on X. Then we let

$$X/\sim$$

and

$$X/E$$

denote the set of equivalence classes of E. That is,

$$X/E = \{A \subset X \text{ such that } A \text{ is an equivalence class.}\}$$

We call X/E the *quotient set* of X (with respect to E).

Remark 9.2.10. So for example, the following notations make sense:

$$A \in X/E, \qquad A \subset X, \qquad x \in A.$$

However, the following *does not* make sense:

$$A \subset X/E, \qquad x \in X/E.$$

Note that we have a function

$$q : X \to X/E, \qquad x \mapsto \text{The equivalence class } A \text{ containing } x.$$

We know that every $x \in X$ belongs to some equivalence class because of property (0) of an equivalence relation, and we know that every x belongs to a unique equivalence class because of the exercise—hence the function q is indeed well-defined.

Definition 9.2.11. The function $q : X \to X/E$ is called the *quotient map* (with respect to E).

Remark 9.2.12. Intuitively, the data of E tells you which points to glue together. X/E is the set one gets after gluing together those points. The quotient map q tells you that a point $x \in X$ goes to the point in X/E resulting from gluing together all those points related to x.

Remark 9.2.13. The function q is a surjection. This is by property (0) of equivalence class: Any equivalence class has at least one element in it, hence any A equals $q(x)$ for some x.

9.2.1 The collection of lines through the origin in $\mathbb{R}^2$

We tackled this set a week ago. Let's give this set a name.

Notation 9.2.14 ($\mathbb{R}P^1$)**.** We let $\mathbb{R}P^1$ denote the set of all lines through the origin in $\mathbb{R}^2$.

Remark 9.2.15. This notation is common in the literature. $\mathbb{R}P^1$ is also called the *real projective line*.

Our goal is to understand whether we can think of $\mathbb{R}P^1$ as a topological space.

How do you *specify* a line through the origin in $\mathbb{R}^2$?

Approach One. Specify a point on the circle. Then there's a unique line that goes through that point and the origin. So there is a function

$$p : S^1 \to \mathbb{R}P^1.$$

Note that this function is not one-to-one; for example, two antipodal points on a circle (i.e., two points given by angle θ and by $\theta\pi$) determine the same line. Regardless, p is a surjection, so we can try to endow $\mathbb{R}P^1$ with the topology you induced on homework.

Approach two. Specifying an equation for the line. Recall that any line can be expressed as the set of pairs x_1, x_2 satisfying the equation

$$ax_1 + bx_2 = c.$$

If the line is to pass through the origin, then we know c must equal zero. Moreover, for the above equation to specify a line, then at least one of a or b

must be non-zero. So any pair $(a, b) \neq (0, 0)$ determines a line $L_{a,b}$ through the origin:

$$(a, b) \mapsto L_{a,b} = \{(x_1, x_2) \text{ such that } ax_1 + bx_2 = 0.$$

This defines another function

$$p : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}P^1, \qquad (a, b) \mapsto L_{a,b}.$$

This is a surjection because every line through the origin is determined by some equation of the form $ax_1 + bx_2 = 0$. However, this function is not an injection.

Exercise 9.2.16. Fix a pair (a, b) and (a', b'). Then $L_{a,b} = L_{a',b'}$ if and only if there exists a non-zero real number $t \neq 0$ such that

$$(ta, tb) = (a', b').$$

So we have laid out two approaches. In either approach, we have found a set X together with a surjection $p : X \to \mathbb{R}P^1$. Moreover, intuitively, both these surjections should feel continuous. (Informally: If you wiggle a point in S^1, you are wiggling the line passing through that point. If you wiggle the parameters a and b, you are wiggling the line given by that parameter.) So a natural way to give a topology to $\mathbb{R}P^1$ is by giving it the *quotient topology* induced by the surjections p (as defined in your homework).

9.3 When are two spaces equivalent?

So we have two distinct ways of exhibiting a surjection to $\mathbb{R}P^1$:

1. As a quotient of the sspace $\{(a, b) \text{ such that } (a, b) \neq (0, 0) = \mathbb{R}^2 \setminus \{(0, 0)\}$, and
2. As a quotient of S^1.

So, a priori, we have two *different* topologies on $\mathbb{R}P^1$. Are they the same? Put another way, are the quotient topologies on

$$(\mathbb{R}^2 \setminus \{(0, 0)\}/\sim) \qquad \text{and} \qquad S^1/\sim$$

"equivalent" in some sense?

This brings us to a natural question:

When should we consider two topological spaces to be equivalent?

I want to emphasize a difference between two things being "equal" (or the same) and two things being "equivalent." For example, a set of three bananas is not the same set as a set of three apples. But they can be treated as equivalent for many set-theoretic purposes. The reason is that they have the same "size," or cardinality; that is, the two sets are in bijection.

Put another way, we consider two sets to be equivalent if there exists a bijection between them. And the bijection exhibits in what way we consider them to be equivalent.

So how about spaces? Spaces are not just sets, but sets equipped with a topology (i.e., a collection open sets). So we should consider two spaces to be equivalent if they are not only equivalent as sets, but also have "equivalent" collections of open sets. More on this next time.

Lecture 10

Thursday, September 26th

What we learned last time has a huge pay-off: We get to construct a lot of fun and interesting spaces.

10.1 Elaborations on last time

Last time we talked about equivalence relations and equivalence classes; these allowed us to construct the quotient topology on quotient spaces. Let me introduce a bit of notation:

Notation 10.1.1. Let E be an equivalence relation on X, and let $A \subset X$ be an equivalence class containing some $x \in X$. We will write

$$[x] \subset X$$

for this equivalence class. So $[x] = [x']$ if and only if $x \sim x'$.

Example 10.1.2. Let $X = \{1, 2, 3, 4\}$ be a set of 4 elements we call 1, 2, 3 and 4. Let's say we want to glue 2 to 3, and 3 to 4. Such a gluing will result in a set with two elements.

Let's be explicit about the equivalence relation $E \subset X \times X$ that encodes the idea that we want to glue 2 to 3 and 3 to 4. E is, explicitly:

$$\begin{aligned} E = \{&(1,1), (2,2), (3,3), (4,4), \\ &(2,3), (3,2) \\ &(3,4), (4,3) \\ &(2,4), (4,2)\}. \end{aligned}$$

The first row of elements are in E by reflexivity: Any element should be related to itself. For example, $(1,1) \in E$ means that $1 \sim 1$.

The second row follows because we want to glue 2 to 3 (so we want $2 \sim 3$) and by symmetry (which says that 3 must be related to 2 — i.e., $(3,2) \in E$).

Likewise for the third row, because we want to glue (i.e., declare equivalent) 3 to 4.

Finally, the last row follows by transitivity: If we are gluing 2 to 3, and 3 to 4, then we are also gluing 2 to 4.

We can list the equivalence classes of E explicitly. We have two:

$$A = \{1\}, \qquad B = \{2,3,4\}.$$

The set $X/E = \{A, B\}$ is the two element-set we obtain by gluing as prescribed.

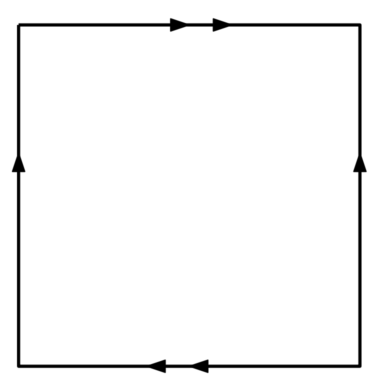
Remark 10.1.3. To reiterate: An equivalence relation E tells you what you elements you want to glue together, and the quotient set X/E is the result of gluing.

Let me motivate quotient spaces a little more. You are probably used to visualizing shapes in three-dimensional space (i.e., $\mathbb{R}^3$); but there are spaces that cannot be visualized as sitting in $\mathbb{R}^3$. Here is an example:

Example 10.1.4. Let $X = [0,1] \times [0,1] \subset \mathbb{R}^2$; X is a square. Consider the relation

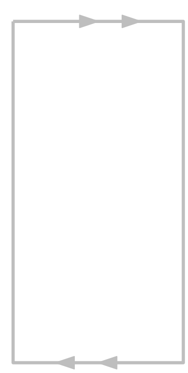
$$(0, x_2) \sim (1, x_2) \qquad \text{and} \qquad (x_1, 0) \sim (1 - x_1, 1)$$

(and of course, (x_1, x_2) is related to itself) for all $x_1, x_2 \in [0,1]$. The common shorthand drawing for this gluing is as follows:



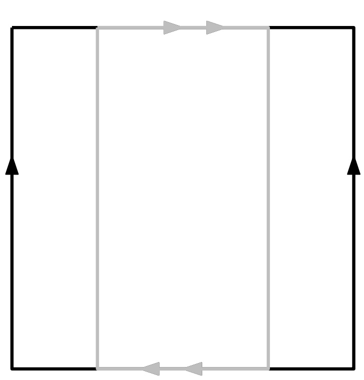
(Note that the vertical edges are glued in a way respecting their orientations, while the two horizontal edges are glued in a way that flips them. You should try and make this shape at home. I promise you won't be able to do it in a way where things are embedded nicely (in fact, embedded at all!) in your three-dimensional space. The upshot is that quotient spaces give us a way of talking about many different kinds of spaces—even those that we cannot visualize in three-dimensions. Note also that (because of our particular presentation) it seems at first difficult to try to embed this space in $\mathbb{R}^n$ for any n.

Remark 10.1.5. However, if we only glue the two horizontal edges (with orientations flipped), one can perform this gluing in our three-dimensional space.



Give it a try if you've never seen this before. The result space is called a *Mobius strip*.

Let me also remark that the mobius strip actually embeds into the shape from the previous example:



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So even though we cannot visualize easily the shape from the previous example, we do know that it contains a mobius strip (somehow).

So at the very least, we can motivate quotient spaces as follows: Quotient spaces are often examples of interesting spaces.

10.2 Coproducts (disjoint unions)

Here is another way:

Notation 10.2.1. Let X and Y be sets. We let

$$X \coprod Y$$

denote the *disjoint union* of X and Y. This is also called the *coproduct* of X and Y.

Remark 10.2.2. In case you haven't see $\coprod$ before, let me tell you how it's different from the usual union operation. For example, with the ordinary union, $X \cup X = X$; that is, any set union itself gives back that set.

But $X \coprod X \neq X$. Disjoint union means we formally treat the two sets as made of distinct elements (even if the sets may have intersection) and then take the union. So for example, if X is a finite set with N elements, then $X \coprod X$ is a finite set with $2N$ elements.

A more formal way to construct the disjoint union of sets is as follows. Fix a set $\mathcal{A}$ and for each $a \in \mathcal{A}$, fix a set X_a. Then the disjoint union of the X_a, denote

$$\coprod_{a \in \mathcal{A}} X_a,$$

is the set consisting of those ordered pairs (a, x) where $a \in \mathcal{A}$ and $x \in X_a$. In particular, when $\mathcal{A}$ is a set consisting of two elements a and b, this defines the disjoint union of two sets:

$$X_a \coprod X_b.$$

Definition 10.2.3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. We endow the disjoint union

$$X \coprod Y$$

with the following topology: A subset $W \subset X \coprod Y$ is open if and only if $W \cap X$ and $W \cap Y$ are both open.

We call this the *coproduct* topology on $X \coprod Y$.

Example 10.2.4. Let $*$ be a set consisting of one element. This has a unique topology—every subset is declared open.

Then $* \coprod *$ is a set consisting of two elements. Every subset of $* \coprod *$ is open.

More generally, for any set $\mathcal{A}$, one can take the coproduct

$$\coprod_{\alpha \in \mathcal{A}} *.$$

This coproduct is a set in bijection with $\mathcal{A}$, and it has a topology such that every subset is open.

Example 10.2.5. Let X, Y, Z be topological spaces. Then a function $f : X \coprod Y \to Z$ is continuous if and only if the associated functions $X \to Z$ and $Y \to Z$ are both continuous.

Put another way, to check whether f is continuous, we do not need to check anything that involves both X and Y at once; informally, this means that X and Y do not have topologies that "talk" to each other; a point $x \in X$ has no "desire" or "knowledge" to be within wiggling room of a point of Y.

10.2.1 Discrete spaces

Definition 10.2.6. Let X be a set. The *discrete topology* on X is the topology for which every subset of X is open.

A topological space equipped with the discrete topology is called a discrete space.

Example 10.2.7. Suppose X and Y are discrete spaces. Then any bijection between them is a homeomorphism.

Example 10.2.8. Any discrete space X is homeomorphic to a coproduct; namely, setting $\mathcal{A} = X$, we have that X is homeomorphic to

$$\coprod_{a \in \mathcal{A}} *.$$

Example 10.2.9. Let $X = \mathbb{R}^n$ and equip X with the discrete metric. Then the associated topology on $\mathbb{R}^n$ is the discrete topology. To see this, note that for any $x \in \mathbb{R}^n$, the open ball of radius $r < 1$ centered at x is just the set $\{x\}$ consisting only of x. Hence any subset of $\mathbb{R}^n$ is open (because any set is a union of its elements).

Remark 10.2.10. You should think of a discrete space as made up of a bunch of "disconnected" points.

For example, continuous maps out of discrete spaces are not interesting from the viewpoint of topology: For any topological space Y, and any discrete space X, any function $f : X \to Y$ is automatically continuous. (This is because the preimage of any $V \subset Y$ is some subset of X, but every subset of X is open!)

Intuitively, a function is not continuous precisely when it doesn't respect some desire for points on X to "stay wiggling near each other." That any function $f : X \to Y$ is continuous means that the points of X have no such wiggling relationship with each other.

10.2.2 The trivial topology

Let X be any set. We saw in the previous section that X admits a topology called the discrete topology; it was in some sense a silly topology because any subset of X was deemed open. There is another silly topology: Declare $\emptyset$ and X to be the only open subsets of X.

Definition 10.2.11. Let X be a set. The *trivial topology* on X is the one for which $\emptyset$ and X are the only open sets.

Example 10.2.12. Let W be any topological space, and let X be a space equipped with the trivial topology. Then any function $W \to X$ is continuous.

This is "dual" to the discrete topology; it was easy to construct continuous functions whose *domains* were discrete; what we see is that it is also easy to construct continuous functions whose codomains have trivial topology.

Example 10.2.13. There are only two sets for which the trivial and discrete topology coincide: The empty set, and the set with one element.

Example 10.2.14. Let X be a set, and let $\mathcal{T}_{triv}$ and $\mathcal{T}_{discrete}$ be the trivial and discrete topologies on X, respectively. Then the identity function

$$(X, \mathcal{T}_{discrete}) \to (X, \mathcal{T}_{triv})$$

is continuous, but the identity function

$$(X, \mathcal{T}_{triv}) \to (X, \mathcal{T}_{discrete})$$

is not.

10.3 Summary of how to make topological spaces

So far we've seen:

1. Metric spaces give rise to topological spaces. Explicitly, given (X, d), we declare $U \in \mathcal{T}_X$ if and only if U is a union of open balls. (Example: $(\mathbb{R}, d_{std})$ gives rise to $\mathbb{R}$ with the standard topology—a subset of $\mathbb{R}$ is open if and only if it is a union of open intervals.)

2. Products of topological spaces have natural topological space structures. (Example: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and more generally, $\mathbb{R}^n$.) Given $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, we declared a subset of $X \times Y$ to be open if and only if it is a union of sets of the form $U \times V$ with $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$.

3. Subsets of topological spaces have natural topological space structures. (Example: $S^1 \subset \mathbb{R}^2$).

4. Quotients of topological spaces have natural topological space structures. (Example: $\mathbb{R}P^1$, or the "cylinder"-like object.)

5. Coproducts

We have so many ways of making new spaces, that maybe you'll do some different things and end up making equivalent spaces! But what does it mean for spaces to be equivalent?

10.4 Homeomorphism

Last class we left off on the topic of: When are two spaces equivalent?

Proposition 10.4.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Fix a bijection $f : X \to Y$. Then the following are equivalent:

1. The preimage operation f^{-1} sends open sets of Y to open sets of X; moreover the induced function $\mathcal{T}_Y \to \mathcal{T}_X$ is a bijection.

2. f is continuous, and the inverse of f is continuous.

I did not give the proof in class, but you may assume this result from now on. Here is a proof for those who are curious:

Proof. (1) $\implies$ (2): By definition, if the operation $V \mapsto f^{-1}(V)$ sends open sets to open sets, then f is continuous. Let g be the inverse function to f. We must show that g is continuous to finish the proof.

If $U \subset X$ is open, we must verify that $g^{-1}(U)$ is open in Y. Well, the operation $f^{-1} : \mathcal{T}_Y \to \mathcal{T}_X$ is a bijection, and in particular, a surjection. Hence there is some open subset $V \subset Y$ such that $U = f^{-1}(V)$. Because g is the inverse to f, we have that $g^{-1}(U) = f(U) = f(f^{-1}(V))$; because f is a surjection, $f(f^{-1}(V)) = V$. Thus g is continuous, as was to be shown.

(2) $\implies$ (1): Because f is continuous, we know that the preimage operation f^{-1} defines a function $\mathcal{T}_Y \to \mathcal{T}_X$. This is an injection because f is a surjection.[1] To show that $\mathcal{T}_Y \to \mathcal{T}_X$ is a surjection, we invoke that the inverse function g is continuous—for then the preimage operation defines a function $g^{-1} : \mathcal{T}_X \to \mathcal{T}_Y$, and because $f^{-1}(g^{-1}(U)) = U$, we conclude that the preimage operation f^{-1} is a surjection. □

So this gives us the following notion of equivalence of topological spaces:

Definition 10.4.2. Let X and Y be topological spaces. We say that a function $f : X \to Y$ is a *homeomorphism* if

1. f is a bijection,

2. f is continuous, and

3. The inverse of f is continuous.

We will say that two topological spaces are *homeomorphic* if there exists a homeomorphism between them, and we will write

$$X \cong Y$$

to mean that X is homeomorphic to Y.

[1] In general, for any function f, we have that $f(f^{-1}(V)) \subset V$. When f is a surjection, $f(f^{-1}(V)) = V$. In particular, if f is a surjection, we have that $f^{-1}(V) = f^{-1}(V') \implies V = V'$.

Example 10.4.3. Note that even if f is a bijection and continuous, it may be that the inverse to f is not continuous. An example is given by the identity function

$$f = \mathrm{id} : (\mathbb{R}^n, d_{discrete}) \to (\mathbb{R}^n, d_{std}).$$

This is a continuous function, but its inverse function (which is again the identity function) is not continuous.

10.5 Examples of quotient spaces

Quotient spaces take some time to get used to; but I want to encourage you to think freely and pictorially about them as you also learn to think rigorously about them. Here are examples to give you some intuition.

Example 10.5.1. Let $X = [0,1]$ be the closed interval from 0 to 1. Let E be the equivalence relation where the only non-trivial relation is $0 \sim 1$. We will write

$$X/(0 \sim 1)$$

for the quotient space (equipped with the quotient topology from your homework).

Then

$$X/(0 \sim 1) \cong S^1.$$

That is, this quotient space is homeomorphic to a circle. This is "clear" if you know how to visualize things, but otherwise it can seem like a non-trivial statement. We'll talk about how to prove this at a later time, but you should draw a picture to see why this might be true.

Lecture 11

More on equivalence relations, and $\mathbb{R}P^n$

11.1 Surjections and equivalence relations

A question was asked in class:

"Does any continuous surjection $p : X \to Y$ form an equivalence relation?"

We recognized this was imprecise because we have no definition for when a function "forms" an equivalence relation. For example:

1. Equivalence relation on what? On X or on Y?

2. What relationship would one like p to have with this relation?

3. What does continuity have to do with it?

Indeed, let's take continuity out of the picture—so that X and Y are just sets, and p is just a function. After several rounds of discussion, we came upon the following question:

Question 11.1.1. Let $p : X \to Y$ be a surjection. Does there exist an equivalence relation on X so that $X/\sim$ is in bijection with Y?

Here is what we saw:

Proposition 11.1.2. Let $p : X \to Y$ be a surjection. And define a relation on X by

$$x \sim x' \iff p(x) = p(x'). \tag{11.1.0.1}$$

Equivalently, this relation is given by the set $E \subset X \times X$ where

$$E = \{(x, x') \text{ such that } p(x) = p(x')\}.$$

Then this is an equivalence relation.

Proof. (Reflexivity.) We must show that for all $x \in X$, we have $x \sim x$. This follows because $p(x) = p(x)$.

(Symmetry.) We must show that for all $x, x' \in X$, if $x \sim x'$, then $x' \sim x$. Well,

$$x \sim x' \implies p(x) = p(x') \implies p(x') = p(x) \implies x' \sim x.$$

(Transitivity.) We must show that for all $x, x', x'' \in X$, if $x \sim x'$ and $x' \sim x''$, then $x \sim x''$. Here is a proof:

$$x \sim x', x' \sim x'' \implies p(x) = p(x'), p(x') = p(x'') \implies p(x) = p(x'') \implies x \sim s''.$$

□

Then we proved:

Proposition 11.1.3. Let $X/\sim$ be the quotient set defined by the equivalence relation (11.1.0.1). Then there exists a bijection from $X/\sim$ to Y.

Remark 11.1.4. Just to make sure we know what's going on:

- The definition of $\sim$ depended on p; we expect the function $X/\sim \to Y$ to also depend in some way on p.
- $X/\sim$ is the set of equivalence classes of $\sim$. that is, it is a set of sets.
- Recall that an equivalence class of $\sim$ is a subset $A \subset X$ such that
  - if $x \in A$, then all x' such that $x' \sim x$ is also in A; moreover,
  - if $x, x' \in A$, then $x \sim x'$
- Recall also that if $x \in A$ and A is an equivalence class, we write

$$[x] = A.$$

Proof. Let's first define the bijection. We will call it $\phi : X/\sim \to Y$.

Given $A \in X/\sim$, let $x \in A$. Then we define

$$\phi(A) := p(x).$$

(Well-definedness of ϕ.) Note that this function ϕ seems to depend on something—that is, to define $\phi(A)$, we first had to *choose* $x \in A$, and then apply p to x. But our function should depend only on A (the element of the domain $X/\sim$) and not on a choice of x. Let us verify this. If we had chosen another $x' \in A$, then—by definition of equivalences class—we know that $x \sim x'$. Hence—by the definition of $\sim$ in (11.1.0.1)—we know $p(x) = p(x')$. So ϕ is *well-defined*.[1] Which is to say, ϕ is indeed a function with the specified domain and codomain.

(Injection.) We now prove ϕ is an injection. This means we must show that if $\phi(A) = \phi(A')$, then $A = A'$.

So suppose $\phi(A) = \phi(A')$. By definition of ϕ, that means that for all $x \in A$ and $x' \in A'$, we have

$$p(x) = \phi(A) = \phi(A') = p(x').$$

But by definition of our equivalence relation (11.1.0.1), we know that $p(x) = p(x') \implies x \sim x'$. So $A = A'$ because two equivalences classes that share an element are identical. (This is from a previous class.)

(Surjection.) We now prove ϕ is a surjection. Fix $y \in Y$. Because p is a surjection, there exists $x \in X$ so that $p(x) = y$. Let $A = [x]$ be the equivalence class containing x. Then by definition of ϕ, we have that

$$\phi(A) = \phi([x]) = p(x) = y.$$

This proves that ϕ is a surjection. $\square$

Remark 11.1.5. The only place we used that p is a surjection is in proving that ϕ is a surjection. In general, regardless of whether p is a surjection, we will always have that $X/\sim$ is in bijection with the image of p.

Next, we can actually try to ask some question about topology. Namely,

[1] In general, we say that an assignment a priori depending on particular choices is *well-defined* if it does not depend on those choices.

Question. Let X and Y be topological spaces, and fix a continuous surjection $p : X \to Y$. Consider the bijection

$$\phi : X/\sim \to Y$$

from above.

1. Is ϕ continuous?

2. Is ϕ a homeomorphism?

Let's take this one step at a time. Recall:

Definition 11.1.6 (Quotient topology.)**.** Let X be a topological space and $\sim$ an equivalence relation on X. Let

$$q : X \to X/\sim \qquad x \mapsto [x]$$

be the quotient map. Then we define a topology on $X/\sim$ by declaring that $U \subset X/\sim$ is open if and only if $q^{-1}(U)$ is open.

Proposition 11.1.7. Give $X/\sim$ the quotient topology. Then the quotient map $q : X \to X/\sim$ is continuous.

Proof. We must prove that for any $U \subset X/\sim$ open, $q^{-1}(U)$ is open. This is how openness for a subset of $X/\sim$ is defined. □

Importantly, note that the quotient topology on $X/\sim$ is *exactly* the topology you put (in homework) on the codomain of any surjection. In particular, we have the following result from homework:

Proposition 11.1.8. Let $f : X/\sim \to Z$ be a function. Then f is continuous if and only if the composition $f \circ q$ is open.

By the way, going back to the question above: While the map $X/\sim \to Y$ is continuous, it is not always a homeomorphism. For example, let X be $\mathbb{R}^n$ with the discrete topology, and Y be $\mathbb{R}^n$ with the standard topology (induced by the standard metric). Then the identity function $X \to Y$ is continuous, and the map $X/\sim \to Y$ is also continuous (and a bijection). But its inverse is not continuous, because the composite

$$\mathrm{id} : Y \to X/\sim \to X$$

is not a continuous map. (Note that because $\mathrm{id} : X \to Y$ is a bijection, the quotient map $X \to X/\sim$ has is a bijection, hence there is an inverse $X/\sim \to X$.)

11.2 $\mathbb{R}P^1$ and $\mathbb{R}P^2$

Next time, we will talk more about the following two spaces:

$\mathbb{R}P^1$, which is the space of all lines through the origin in $\mathbb{R}^2$. This is topologized by noticing that there is a surjection

$$p : S^1 \to \mathbb{R}P^1$$

which sends a point x on the circle to the unique line passing through x and the origin. p is a surjection (because any line through the origin intersects the circle at some x), and is not an injection, but is a 2-to-1 map (every line through the origin goes through exactly two points on the circle, so for every $L \in \mathbb{R}P^1$, there are exactly two points in $p^{-1}(L)$). Then we can endow $\mathbb{R}P^1$ with the quotient topology.

Likewise, let $\mathbb{R}P^2$ be the space of all lines through the origin in $\mathbb{R}^3$. How is this a space? That is, how do we topologize it?

We do the same trick as before: We notice there is a function $S^2 \to \mathbb{R}P^2$. Given a point x on the sphere, there is a unique line through the origin that also passes through x. We call this assignment $p : S^2 \to \mathbb{R}P^2$. Then p, as before, is a 2-to-1 surjection. We topologize $\mathbb{R}P^2$ by the quotient topology.

This $\mathbb{R}P^2$ is a cool space. It turns out it cannot be embedded into $\mathbb{R}^3$, so we do not have a perfect way of visualizing it. Moreover, we will eventually see that $\mathbb{R}P^2$ admits an embedding of the Mobius band inside of it.

More on this next time.

Lecture 12

Real projective plane

12.1 $\mathbb{R}P^1$ and $\mathbb{R}P^2$

Recall that $\mathbb{R}P^1$ is the set of lines through the origin in $\mathbb{R}^2$. And $\mathbb{R}P^2$ is the set of lines through the origin in $\mathbb{R}^3$. These spaces are pronounced "R P one" and "R P two," respectively.

Remark 12.1.1. Somebody asked what this notation stands for.

$\mathbb{R}$ stands for the real numbers.

P stands for the word "projective." This word originates in "projective geometry," which is the study of how the geometry of our world behaves when it's projected onto (for example) a canvas, or our retina.

Sometimes, $\mathbb{R}P^1$ is called the *real projective line*, and $\mathbb{R}P^2$ is called the *real projective space*.

Remark 12.1.2. Somebody asked if there is a "complex" version, say $\mathbb{C}P^1$ and $\mathbb{C}P^2$. There are such things. Recall that we have seen that $\mathbb{R}P^1$ can be written as the following quotient set:

$$\{(x_1, x_2) \neq (0,0)\}/\sim, \qquad (x_1, x_2) \sim (x_1', x_2') \iff x_1 = tx_1' \text{ and } x_2 = tx_2' \text{ for some } t \neq 0.$$

Well, we can now pretend that x_1, x_2 are *complex* number, and define a quotient set using the exact same notation as above (with t now also a complex number). This is a quotient of the space $\mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$. Since $\mathbb{C} \cong \mathbb{R}^2$, this is a quotient of the space $\mathbb{R}^4 \setminus \{(0,0,0,0)\}$. We call this quotient $\mathbb{C}P^1$.

It turns out that $\mathbb{C}P^1$ is homeomorphic to the sphere, S^2.

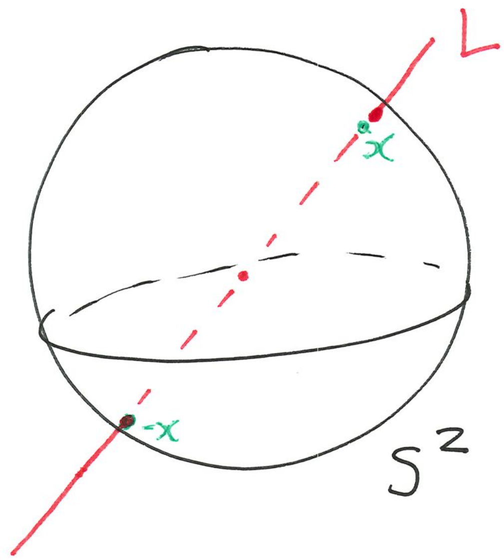
12.2 The topology of $\mathbb{R}P^2$

Today we're going to study $\mathbb{R}P^2$. It's a great space.

Recall that we have defined a function

$$p : S^2 \to \mathbb{R}P^2$$

from the sphere to $\mathbb{R}P^2$. It sends a point $x \in S^2$ to the unique line L passing through x and the origin.



p is a surjection because every line through the origin passes through some point on the sphere.

p is not an injection. Indeed, every line through the origin passes through *two* points on the sphere. Thus p is a *two-to-one* map, meaning that every point in the codomain has a preimage of size two.

Remark 12.2.1. Fix a line $L \in \mathbb{R}P^2$. Note that if x is a point in $L \cap S^2$, then the point $-x$, defined by

$$x = (x_1, x_2, x_3) \implies -x = (-x_1, -x_2, -x_3)$$

is the other point in $L \cap S^2$. So we see that $p^{-1}(L) = \{x, -x\}$.

Definition 12.2.2 (Quotient topology, I. This is from homework.)**.** Let X be a topological space. If $p : X \to Y$ is a surjection, we topologize Y as follows: A subset $V \subset Y$ is open if and only if $p^{-1}(V)$ is open in X.

On the other hand, we have:

Definition 12.2.3 (Quotient topology, II.)**.** Let X be a topological space and $\sim$ an equivalence relation on X. Let $q : X \to X/\sim$ be the quotient map—i.e., $q(x) = [x]$. We topologize $X/\sim$ so that $V \subset X/\sim$ is open if and only if $q^{-1}(V)$ is open.

Remark 12.2.4. Note that the second definition is a special case of the first, because $q : X \to X/\sim$ is always a surjection.

Now we put the two definitions together. Let $p : X \to Y$ be a surjection. Recall from last week that p defines an equivalence relation $\sim$ on X given by $x \sim x' \iff p(x) = p(x')$, and that we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "q"'] & & Y \\ & X/\sim . \arrow[ur, "\phi"'] & \end{tikzcd}$$

Here, ϕ is a function given by $\phi([x]) = p(x)$. That the diagram is commutative means that $p = \phi \circ q$. Moreover, we also saw last week that ϕ is a bijection.

If we give $X/\sim$ the quotient topology (II), then there is a *unique* topology on Y so that ϕ is not only a bijection, but a homeomorphism. This is the topology for which $V \subset Y$ is open if and only if $\phi^{-1}(V)$ is open in $X/\sim$.

Definition 12.2.5 (Quotient topology, III.)**.** Let $p : X \to Y$ be a surjection, and let $\sim$ be the equivalence relation $x \sim x' \iff p(x) = p(x')$. Then we topologize Y so that $V \subset Y$ is open if and only if $\phi^{-1}(V)$ is open in $X/\sim$.

We leave the following for you to verify:

Proposition 12.2.6. The definitions I and III yield the same topology on Y.

Definition 12.2.7. Let X be a topological space and let $p : X \to Y$ be a surjection. The topology of Definition I (or III) is called the *quotient topology* on Y.

Definition 12.2.8. Consider the surjection $p : S^2 \to \mathbb{R}P^2$ discussed above. We topologize $\mathbb{R}P^2$ using the quotient topology.

Remark 12.2.9. So we are using two of our "how to make a new space" constructions to define $\mathbb{R}P^2$. First, note that we have topologized S^2 by the subspace topology—$\mathbb{R}^3$ has a standard topology, and we give S^2 the subspace topology. Second, we have used the quotient space construction.

12.3 An open subset

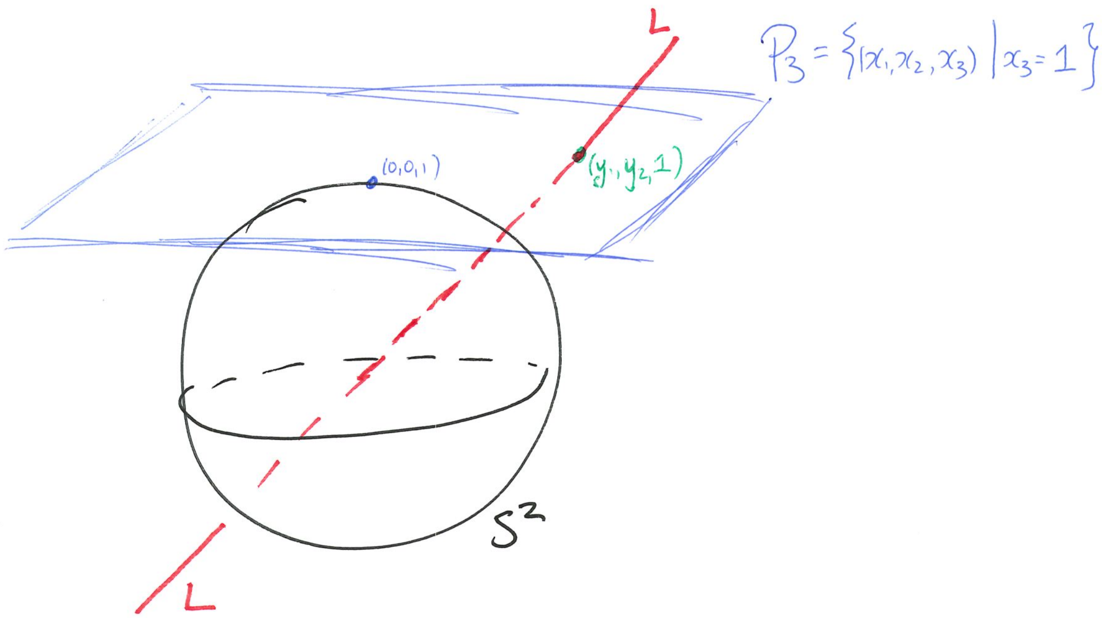
I would like to better understand $\mathbb{R}P^2$. So we're going to try to start understanding subsets of $\mathbb{R}P^2$ in terms of spaces I understand. Well, the only space that is remotely familiar to me is $\mathbb{R}^2$. So can I construct functions from $\mathbb{R}P^2$ to $\mathbb{R}^2$, and perhaps vice versa, that will help me understand $\mathbb{R}P^2$?

Here is a fun construction. Let $L \in \mathbb{R}P^2$ be a line through the origin. And fix a plane P_3 given by the equation $x_3 = 1$. Concretely,

$$P_3 = \{(x_1, x_2, x_3) \text{ such that } x_3 = 1\} \subset \mathbb{R}^3.$$

Then, if L intersects P_3, we have a unique intersection point $y \in L \cap P_3$. If we call its coordinates y_1, y_2, y_3, we know $y_3 = 1$, so we may as well only remember the pair (y_1, y_2). This yields an assignment

$$L \mapsto (y_1, y_2).$$



That is, it seems we almost have a function from $\mathbb{R}P^2$ to $\mathbb{R}^2$.

I say *almost* because not every line $L \in \mathbb{R}P^2$ intersections P_3. Indeed, what if L is the line given by the x_1 or the x_2 axis? In general, if L is *parallel* to the plane P_3, then L never intersects P_3, and we have no way of producing the numbers (y_1, y_2).

So this geometric construction doesn't produce a function from $\mathbb{R}P^2$ to $\mathbb{R}^2$, but it does produce a function from a *subset* of $\mathbb{R}P^2$ to $\mathbb{R}^2$. Let's give this subset a name.

Notation 12.3.1. Let $U_3 \subset \mathbb{R}P^2$ the set of those lines that intersect P_3.

Then the above construction defines a function

$$j_3 : U_3 \to \mathbb{R}^2, \qquad L \mapsto (y_1, y_2)$$

where $(y_1, y_2, 1)$ is the unique point in $L \cap P_3$.

Here is the big question of the day: Is U_3 open?

12.4 Proving U_3 is open

This is a great exercise in all the definitions.

12.4.1 Using the definition of quotient topology to reduce the problem to a subset of S^2

By definition, $U_3 \subset \mathbb{R}P^2$ is open if and only if its preimage in S^2 is open (its preimage under the map $p : S^2 \to \mathbb{R}P^2$).

Remember that p is the map sending a point x to the line passing through x and the origin. As such, the preimage of U_3 is the set of those $x \in S^2$ such that the line through x and the origin also passes through the plane P_3.

But given x, the line through x and the origin intersects P_3 if and only if the coordinate x_3 of x is non-zero. Thus, we find

$$p^{-1}(U_3) = \{x = (x_1, x_2, x_3) \in S^2 \text{ such that } x_3 \neq 0\}.$$

Let us call this set V_3.

Thus, to see whether U_3 is open, we must test whether V_3 is an open subset of S^2.

12.4.2 Using the definition of subset topology to reduce the problem to a subset of $\mathbb{R}^3$

Recall that S^2 is given the *subspace* topology as a subspace of $\mathbb{R}^3$. By definition, a subset $V \subset S^2$ is open if and only if there exists an open $W \subset RR^3$ for which

$$V = W \cap S^2.$$

Just as an ϵ-δ proof requires you to produce a δ given an ϵ, we must now exhibit a W given a V to prove that V is open.

Our V in question is the set $V_3 \subset S^2$ be the set of those $x \in S^2$ whose x_3 coordinate is non-zero. I claim V_3 is open.

So what is the open set $W \subset \mathbb{R}^3$?

Proposition 12.4.1. Let $W \subset RR^3$ denote the set of all elements $x \in \mathbb{R}^3$ for which $x_3 \neq 0$. Then :

(1) W is open in $\mathbb{R}^3$.

(2) $W \cap S^2 = V_3$.

Proof. We will omit the proof of (2). If (2) is not clear to you, just carefully think about the definitions.

To prove (1), it suffices to prove that any $x \in W$ is contained in some open ball $B(x, r)$ such that $B(x, r) \subset W$.[1]

Well, given $x \in W$, we know that x has distance $r = |x_3|$ from the plane $\{x_3 = 0\}$. (This is otherwise known as the x_1-x_2 plane.) Thus any element $x' \in \mathbb{R}^3$ of distance less than r is also contained in W. We conclude that $B(x, |x_3|) \subset W$, so W is open. $\square$

Using the proposition, we conclude that $V_3 \subset S^2$ is an open subset. Because $V_3 = p^{-1}(U_3)$, we further conclude that $U_3 \subset \mathbb{R}P^2$ is an open subset.

12.5 Another open subset

So we have produced an open subset $U_3 \subset \mathbb{R}P^2$, and a function

$$j_3 : U_3 \to \mathbb{R}^2$$

which sends a line L intersecting the plane $P_3 = \{x_3 = 1\}$ to the the first two coordinates of the intersection point $L \cap P_3$.

Note that we did not need to choose the x_3 coordinate. For example, if we had chosen the x_2 coordinate, we could intersect lines with the plane $P_2 = \{x_2 = 1\}$. As before, we see that note every $L \in \mathbb{R}P^2$ intersects P_2; so let $U_2 \subset \mathbb{R}P^2$ be the set of those lines that intersect P_2.

We then have a function

$$j_2 : U_2 \to \mathbb{R}^2$$

given by sending a line L to the pair (y_1, y_3) where $(y_1, 1, y_3)$ is the intersection point of L with P_2. As before, we see that U_2 is an open subset of $\mathbb{R}P^2$.

12.6 A cover

Question: Do U_2 and U_3 cover $\mathbb{R}P^2$? That is,

Does $U_2 \cup U_3$ equal $\mathbb{R}P^2$?

Parsing the definitions, we see that the union $U_2 \cup U_3$ consists of those lines L which pass through at least one of P_2 or P_3.

[1] We saw in a previous class that a subset W of a metric space is open if and only if for every $x \in W$, there is some open ball of positive radius containing x and contained in W.

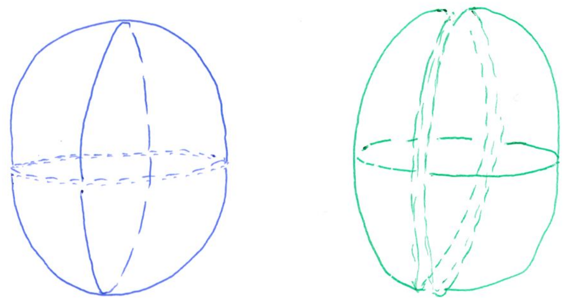


Figure 12.1: Open subsets V_3 (in blue) and V_2 (in green) of S^2.

Then the answer to the question is no. For example, if $U_2 \cup U_3$ were to equal $\mathbb{R}P^2$, then their preimages $V_2 = p^{-1}(U_2)$ and $V_3 = p^{-1}(U_3)$ would have the property that $V_2 \cup V_3 = S^2$, because p is a surjection. But indeed, $V_2 \cup V_3$ is missing exactly two points of the sphere: $(\pm 1, 0, 0)$.

To see this directly from the "set of lines" definitions, note that there is a line, called the x_1-axis, which does not pass through the plane P_2, nor the plane P_3. Indeed, this is the *only* line that does not pass through either of the planes. (Any other line would have a point with either the x_2 or x_3 coordinate being non-zero; in particular, such a line would intersect the plane P_2 (if $x_2 \neq 0$) or P_3 (if $x_3 \neq 0$).)

That is, $U_2 \cup U_3$ is equal to $\mathbb{R}P^2$ with one point removed.

But I want *all* of $\mathbb{R}P^2$.

Well, there is a notationally suggestive thing we can do: Let's define $U_1 \subset \mathbb{R}P^2$ to consist of those lines that pass through the plane $P_1 = \{x_1 = 1\}$. (This P_1 is the plane consisting of those vectors whose x_1 coordinate is equal to 1.) As before, we see that U_1 is open. It also clearly contains the x_1-axis. To summarize, we have:

Proposition 12.6.1. For $i = 0, 1$, or 2, let

$$P_i \subset \mathbb{R}^3$$

denote the set of those points whose x_ith coordinate is equal to 1. We let

$$U_i \subset \mathbb{R}P^2$$

consist of those lines L such that $L \cap P_i$ is non-empty. Then

1. each U_i is an open subset of $\mathbb{R}P^2$. Moreover,
2. The union
$$U_1 \cup U_2 \cup U_3$$
is equal to $\mathbb{R}P^2$.

We will study these open sets more next time.

Lecture 13

Understanding $\mathbb{R}P^2$ more

Most of today was spent going over Homework 6. But it's a long problem. Solutions are posted online.

Recall that last time, we defined three open subsets

$$U_1, U_2, U_3$$

of $\mathbb{R}P^2$.

Here, $U_i \subset \mathbb{R}P^2$ is the collection of all lines L such that L intersects the plane

$$P_i = \{x_i = 1\}.$$

(This notation is shorthand for the set of all points $x = (x_1, x_2, x_3)$ for which $x_i = 1$.)

We saw last time that each $U_i \subset \mathbb{R}P^2$ is open, and that $U_1 \cup U_2 \cup U_3 = \mathbb{R}P^2$.

13.1 Each U_i is a copy of $\mathbb{R}^2$

Now, suppose that $L \in U_3$. Then L intersects the plane P_3 (i.e., the plane of points whose 3rd coordinate is 1). So we can write

$$L \cap P_3 = \{(y_1, y_2, 1)\}$$

where $(y_1, y_2, 1)$ is the unique point in the intersection $L \cap P_3$. This defines for us a function

$$j_3 : U_3 \to \mathbb{R}^2, \qquad L \mapsto (y_1, y_2)$$

where (y_1, y_2) are the first two coordinates of the intersection point $L \cap P_3$.

It turns out that j_3 is continuous. Informally, this is because if we have some open ball around (y_1, y_2), then its preimage will consist of all lines that are close enough to L.

In fact, this is true for any of the U_i. More precisely, note that we have a function

$$j_1 : U_1 \to \mathbb{R}^2, \qquad L \mapsto (y_2, y_3)$$

where $(1, y_2, y_3)$ is the unique intersection point $L \cap P_1$. We also have a function

$$j_2 : U_2 \to \mathbb{R}^2, \qquad L \mapsto (y_1, y_3).$$

More is true:

Proposition 13.1.1. For every $i = 1, 2, 3$ the function

$$j_i : U_i \to \mathbb{R}^2$$

is a homeomorphism.

13.2 Paper mache

Now recall that homeomorphism is our notion of equivalence for topological spaces. So, informally, each time we see a U_i, we can replace it with $\mathbb{R}^2$.

On the other hand, we know that $\mathbb{R}P^2$ is a union of U_1, U_2 and U_3; because each U_i is homeomorphic to $\mathbb{R}^2$, this means that $\mathbb{R}P^2$ is actually a union of three spaces that are all copies of $\mathbb{R}^2$.

Informally, this means we can make $\mathbb{R}P^2$ out of some sort of paper mache—we put together three sheets of paper, but overlapping in some clever way, to make $\mathbb{R}P^2$.

Lecture 14

Atlases and transition functions for $\mathbb{R}P^2$

14.1 The earth is a sphere

I asked a question: How do you know that the earth's surface is (roughly) a sphere?

We were given many great ideas; one of the most convincing was to go into outer space and take a bunch of pictures. All others suffered from being based on taking local measurements, then assuming that some principle allowed us to conclude that those local measurements were valid anywhere on earth. The problem: We can't know that something about Point A on earth is true at Point B.

I claimed that the cheapest way to conclude that the earth is (roughly) a sphere is as follows: Go get an atlas of the earth. Rip out all the pages. Now glue the pages together along their overlaps. (For example, if Page 10 contains Lagos, and Page 33 does, too, then you should glue Page 10 and Page 33 together along where Lagos is displayed.) You are making a very complicated paper mache. And I claim that, in the end, you will end up with something that is roughly spherical.

(There are some issues: You have to assume that the atlas is correct. And indeed, to make a statement about something as large as the earth, you do need to rely on the accuracy of others' knowledge. Another issue is scaling; the scale of Page 10 may not equal the scale of Page 13; so you may have to find an atlas whose pages are made of rubber.)

If you take each page and remove its boundary edges, each page is homeomorphic to $\mathbb{R}^2$ (e.g., to an open rectangle). And each boundary-removed page is then an open subset of the surface of the earth (i.e., of the sphere). What you have just imagined is a procedure of finding a bunch of subsets of S^2 that are all homeomorphic to $\mathbb{R}^2$, and then writing S^2 as a union of these subsets. The way the subsets overlap tells you how to put together the paper mache.

14.2 Paper mache for $\mathbb{R}P^2$

So let's do this for $\mathbb{R}P^2$.

Recall from the last classes:

1. We define $U_1 \subset \mathbb{R}P^2$ to be the set of lines that intersect the plane $P_1 = \{x = (x_1, x_2, x_3) \text{ such that } x_1 = 1\}$. We have seen that this is an open subset, and I have told you that it is homeomorphic to $\mathbb{R}^2$.

   Likewise, we have open subsets U_2 and U_3 of $\mathbb{R}P^2$. Each of these is also homeomorphic to $\mathbb{R}P^2$. We have also seen that

$$U_1 \cup U_2 \cup U_3 = \mathbb{R}P^2.$$

2. Moreover, the homeomorphisms from the U_i to $\mathbb{R}^2$ is given by functions

$$j_i : U_i \to \mathbb{R}^2.$$

   Let us recall how these were defined. Given a line $L \in U_1$, so that L intersects the plane P_1, we can write the intersection point of L and P_1 as follows:

$$(1, y_2, y_3).$$

   The function j_1 sends L to the pair of numbers (y_2, y_3).

   Likewise,

   $j_2(L) = (y_1, y_3)$ when $L \in U_2$, and $\qquad j_3(L) = (y_1, y_2)$ when $L \in U_3$.

14.2.1

Now onto some new material.

Because $U_1 \cup U_2 \cup U_3 = \mathbb{R}P^2$, we see that the induced map

$$h : U_1 \coprod U_2 \coprod U_3 \to \mathbb{R}P^2$$

is a surjection. (Note that the domain here is the coproduct of U_1, U_2, and U_3.) In particular, there exists an equivalence relation $\sim$ on $U_1 \coprod U_2 \coprod U_3$ such that we have an induced bijection

$$(U_1 \coprod U_2 \coprod U_3)/\sim \cong \mathbb{R}P^2.$$

This equivalence relation is one we've seen before: We declare

$$L \sim L' \iff h(L) = h(L').$$

(In general, when we have a surjection $h : X \to Y$, we can define a relation $x \sim x' \iff h(x) = h(x')$ so that we have an induced bijection $X/\sim \to Y$.)

Moreover, because each $U_i \subset \mathbb{R}P^2$ is open, we have:

Proposition 14.2.1. The induced map

$$(U_1 \coprod U_2 \coprod U_3)/\sim \to \mathbb{R}P^2$$

is a homeomorphism.

Now, because each $j_i : U_i \cong \mathbb{R}^2$ is a homeomorphism, we can begin to understand what $\mathbb{R}P^2$ looks like using coordinates on $\mathbb{R}^2$. For example, by using the three homeomorphisms, we have a single homeomorphism

$$j : U_1 \coprod U_2 \coprod U_3 \cong \mathbb{R}^2 \coprod \mathbb{R}^2 \coprod \mathbb{R}^2.$$

Thus we can try to understand the equivalence relation on the lefthandside in terms of the righthand side. That is, the homeomorphism j induces an equivalence relation on the right. What is this relation?

To see what it is, consider the functions

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[d, "j_1^{-1}"'] & & \mathbb{R}^2 \arrow[d, "j_2^{-1}"] \\
U_1 \arrow[r, "\iota_1"] & \mathbb{R}P^2 & U_2. \arrow[l, "\iota_2"']
\end{tikzcd}$$

Then an element $y \in \mathbb{R}^2$ on the left is related to an element $y' \in \mathbb{R}^2$ on the right if and only if

$$\iota_1 \circ j_1^{-1}(y) = \iota_2 \circ j_2^{-1}(y').$$

I claim there is a formula now expressing y in terms of y'. To see this, note that the above equality means that $j_1^{-1}(y)$ and $j_2^{-1}(y')$ must describe the same line (i.e., the same element in $\mathbb{R}^2$). But j_1^{-1} takes the point $y = (y_2, y_3) \in \mathbb{R}^2$ and sends it to the line passing through

$$(1, y_2, y_3).$$

Likewise, j_2^{-1} takes the point $y' = (y_1', y_3') \in \mathbb{R}^2$ and sends it to the line passing through

$$(y_1', 1, y_3').$$

If these points are to be on the same line, then there must be a non-zero real number t so that

$$t(1, y_2, y_3) = (y_1', 1, y_3').$$

That is,

$$t = y_1', \qquad ty_2 = 1, \qquad ty_3 = y_3'.$$

From this, we quickly conclude that for the points $(1, y_2, y_3)$ and $(y_1', 1, y_3')$ to be on the same line L, we must have

$$y_2 = 1/y_1', \qquad y_3 = y_3'/y_1'.$$

Or, equivalently,

$$y_1' = 1/y_2, \qquad y_3' = y_3/y_2.$$

So this gives part of the relation; we see that y and y' are related if and only if the above equations hold. (In particular, y_2 must be non-zero for y to be related to some y', and y_1' must be non-zero for y' to be related to some y.)

Lecture 15

Exam!

Lecture 16

Closed sets and open covers

Class today will have three parts. As I mentioned last week, we're starting a proof bootcamp.

This means every day, you will see new definitions. Then you will spend most of class trying to prove something using the new definitions.

16.1 Closed sets

Definition 16.1.1. Let X be a topological space. A subset $A \subset X$ is called *closed* if and only if its complement is open.

I want you to prove the following in groups:

Proposition 16.1.2. Let $(X, \mathcal{T})$ be a topological space. For this problem, we let $\mathcal{K}$ denote the collection of closed subsets. Show the following are true:

1. $\emptyset, X \in \mathcal{K}$.

2. If $A_1, \ldots, A_n \in \mathcal{K}$ is a finite collection, then $\bigcup_{i=1,\ldots,n} A_i$ is in $\mathcal{K}$.

3. For an arbitrary collection $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ of elements of $\mathcal{K}$, we have that the intersection $\bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is also in $\mathcal{K}$.

Proposition 16.1.3. Let $f : X \to Y$ be a continuous map of topological spaces. Show that if $A \subset Y$ is closed, then its preimage is closed.

Conversely, suppose that $f : X \to Y$ is a function such that whenever $A \subset Y$ is closed, its preimage is closed. Prove that f is continuous.

Proposition 16.1.4. Let $B \subset X$ be an arbitrary subset. Show that there exists a subset, $\overline{B} \subset X$, satisfying the following properties:

1. $B \subset \overline{B}$
2. $\overline{B}$ is closed.
3. Moreover, if C is any other closed subset of X containing B, then C contains $\overline{B}$.

Informally, this means that $\overline{B}$ is the "smallest" closed subset of X containing B.

Definition 16.1.5 (For future use)**.** $\overline{B}$ is called the *closure* of B.

Remark 16.1.6 (Motivation for closed sets)**.** Note that the set of closed sets of a space can automatically recover the set of open sets of a space. (This is because $K \subset X$ is closed if and only if its complement is open.) If you expound upon Proposition 16.1.2, you will see that you can equivalently define a topological space through its closed sets, so long as the collection $\mathcal{K}$ of closed sets of X satisfy all the properties in the proposition. It is then an exercise to show that any such collection $\mathcal{K}$ determines a topology $\mathcal{T}$ (by taking the opens to be complements of elements of $\mathcal{K}$).

The next proposition tells you that you can also equivalently define the notion of continuity through mentioning only closed sets.

One thing you can do freely with closed sets is take intersections, as you saw in Proposition 16.1.2. This allows you to do constructions with closed sets you can't do with open sets. For example, one can convert any set into a closed by taking its closure (which you saw in Proposition 16.1.4). This, informally, gives you a "slightly larger" but closed subset.

In contrast, given some subset $B \subset X$ of a topological space, it is almost impossible to construct a "smallest open set" containing B; you can rather construct the "largest open set" contained in B, and this is called the interior of B. Can you construct it?

16.2 Open covers

Definition 16.2.1. Let $(X, \mathcal{T}_X)$ be a topological space. We say that a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets of X is a *cover* if $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = X$. We further say this collection is an *open cover* if each U_α is open.

I want you to prove the following:

Proposition 16.2.2. Let $\{U_\alpha\}$ be an open cover of X. Note there is a function

$$p : \coprod_{\alpha \in \mathcal{A}} U_\alpha \to X.$$

Prove that the induced map

$$(\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \to X$$

is a homeomorphism. (Here, the equivalence relation $\sim$ is the one for which $x \sim x' \iff p(x) = p(x')$.

Remark 16.2.3 (Motivation for open covers)**.** Proposition 16.2.2 says that you can reconstruct a space X from an open cover of X. This is something special to *open* covers.

For example, given an arbitrary cover $\{A_\alpha\}_{\alpha \in \mathcal{A}}$, even if $\bigcup_{\alpha \in \mathcal{A}} A_\alpha = X$, it need not be true that

$$\left(\coprod_{\alpha \in \mathcal{A}} A_\alpha\right)/\sim \to X$$

is a homeomorphism. For example, you could take $\mathcal{A} = X$ and $A_x = \{x\}$. Then the above map is a homeomorphism if and only if X has the discrete topology.

16.3 The Mobius band in $\mathbb{R}P^2$

In the exam I asked you to convince me there is a Möbius band inside the union $U_1 \cup U_2 \subset \mathbb{R}P^2$.

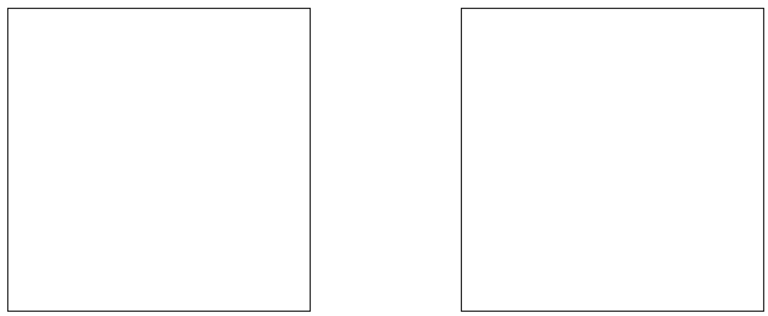
Recall that $U_1 \cup U_2$ is homeomorphic to the following:

$$(\mathbb{R}^2 \coprod \mathbb{R}^2)/\sim$$

where the relation $\sim$ says $y \sim y' \iff$

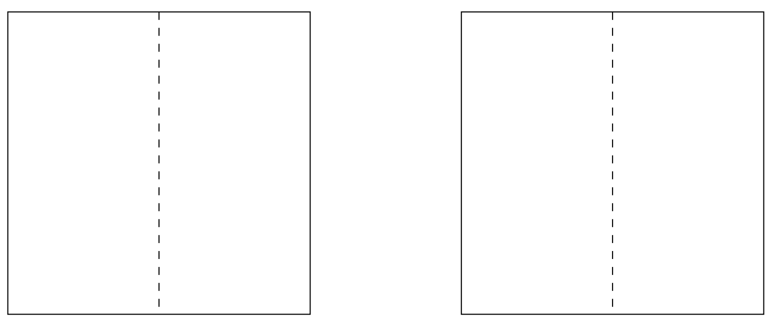
$$\begin{cases} y_2 = 1/y'_1 \text{ and } y_3 = y'_3/y_1, & y \text{ is in the first copy of } \mathbb{R}^2 \text{ while } y' \text{ is in the second copy.} \\ y = y' & y \text{ and } y' \text{ are both in the first copy of } \mathbb{R}^2 \\ y' = y & y \text{ and } y' \text{ are both in the second copy of } \mathbb{R}^2 \\ y'_2 = 1/y_1 \text{ and } y'_3 = y_3/y_1, & y \text{ is in the second copy of } \mathbb{R}^2 \text{ while } y' \text{ is in the first copy.} \end{cases}$$

The relation looks more complicated than it needs to; if you are willing, you can simply think of $\sim$ as the smallest equivalent relation possible containing the first line above.



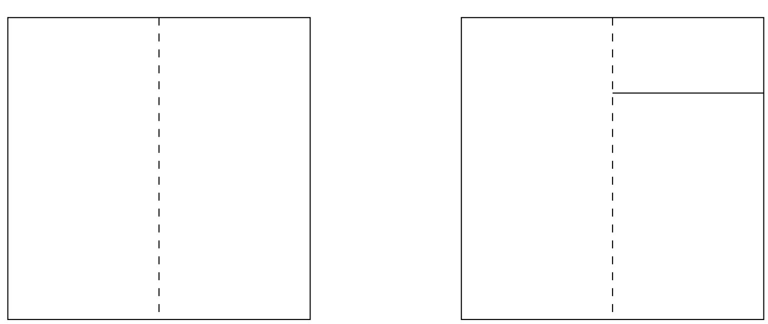
$\mathbb{R}^2 = \{(y_2, y_3)\}$ $\qquad$ $\mathbb{R}^2 = \{(y_1', y_3')\}$

Above is a picture of two copies of $\mathbb{R}^2$.



$\mathbb{R}^2$ $\qquad$ $\mathbb{R}^2$

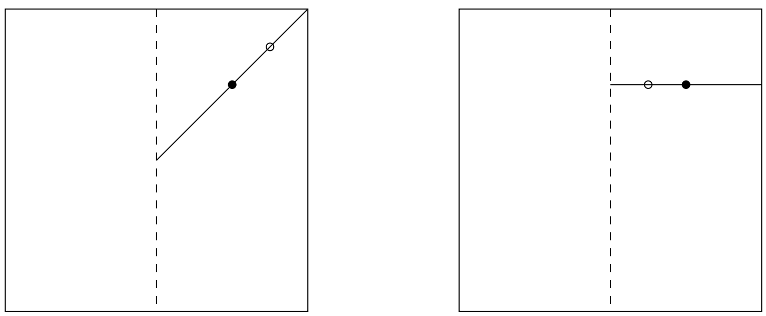
We have drawn (dashed) the lines $y_2 = 0$ (in the lefthand copy of $\mathbb{R}^2$) and the $y_1' = 0$ (in the righthand copy of $\mathbb{R}^2$). I draw these because these points are only related to themselves; they undergo no gluing.



On the right we have drawn (a portion of) the horizontal line $y_3' = a$ for some positive real number a. We have only drawn the portion where $y_1' > 0$. What points on the right are points on $\{y_3' = a\}$ related to? Well, we know

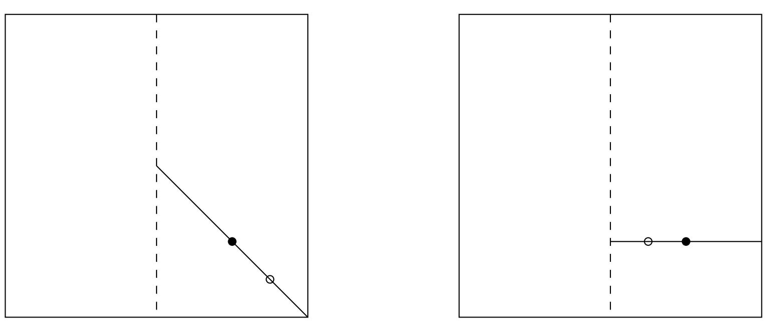
$$y \sim y' \iff y_2 = 1/y_1', y_3 = y_3'/y_1'.$$

In other words, a point of the form $(y_1', y_3') = (y_1', 1)$ on the right is related to a point of the form $(y_2, y_3) = (1/y_1', 1/y_1')$ on the left. These are points where the y_2 and y_3 coordinates are equal; i.e., this is some part of a line! Let's draw the portion where $y_1' > 0$:

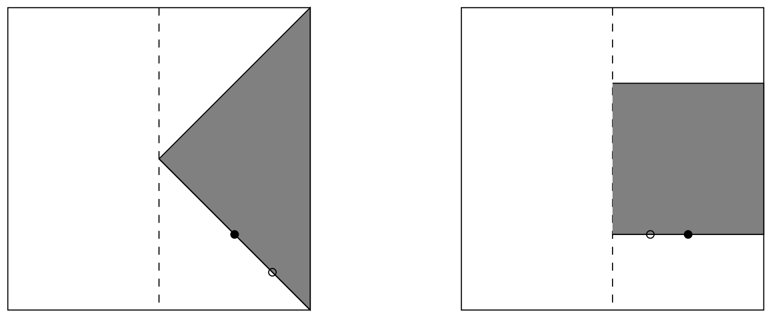


Note, importantly (see the white dot) that when the y_1' coordinate shrinks toward 0, the y_2 coordinate on the right *increases*.

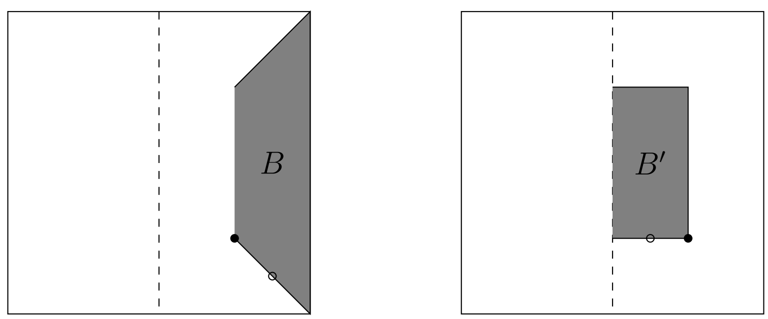
We can likewise draw how the ray $y_3' = -1$, with y_1' positive, is related to a ray in the lefthand side, by reasoning that $(y_1', -1)$ on the right is related to $(1/y_1', -1/y_1')$ on the left. (Thus, points on this horizontal ray on the right, are related to points on a line of slope -1 on the left.)



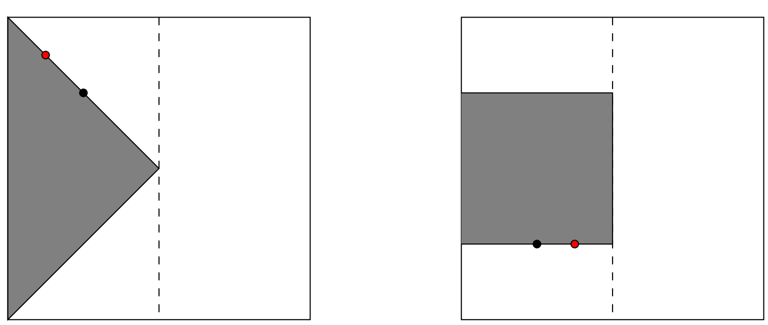
All told, we see that the shaded regions are related to each other as follows:



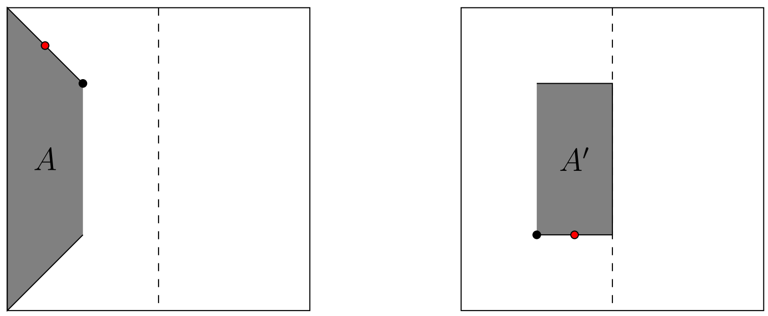
For reasons that will become clear later, let's just remember the shaded region B' on the right, and the shaded region B on the left:



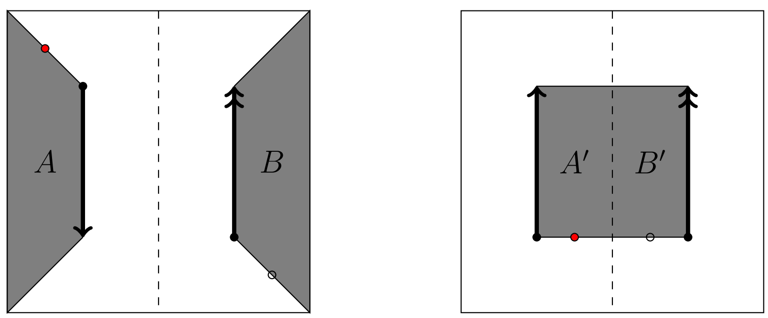
Now I leave it to you to explore what happens when the y_1' coordinate on the righthand copy of $\mathbb{R}^2$ is *negative*, which we haven't considered yet. I claim you'll get the following picture (where points on the shaded region are related to each other in a way I want you to figure out):



Note importantly that though the y_3' coordinates of the dots on the righthand side were negative, the y_3 coordinate of the related points on the righthand side are *positive*! Now consider the regions A and A' indicated below:

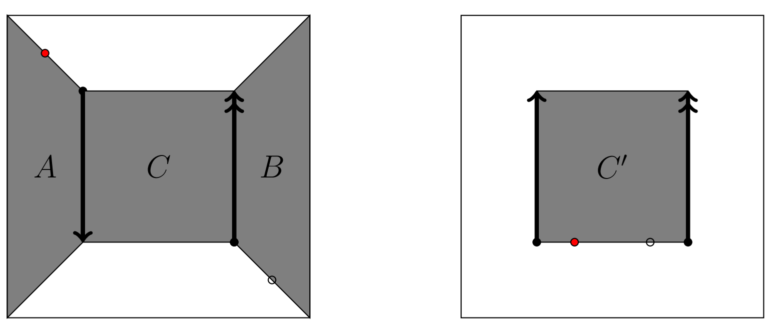


For your convenience, let me re-draw the regions $A, B \subset \mathbb{R}^2$ and $A', B' \subset \mathbb{R}^2$:



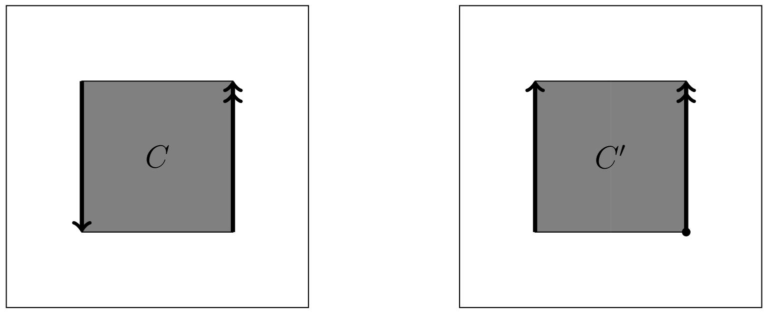
I want to emphasize that A' and B' do not touch; they share no intersection! (The dashed line is important.)

So finally I am ready to draw the Mobius band inside of $U_1 \cup U_2$. Consider the regions C and C' below:



For example, C' contains both A' and B', and a little bit more–it contains some points with $y_1' = 0$, for example.

While C is a subset of (the left copy of) $\mathbb{R}^2$, and C' is a subset of (the right copy of) $\mathbb{R}^2$, they are mapped to a $U_1 \cup U_2$ in a way such that they overlap along the arrowed edges. The overlap is interesting; as indicated, the left edge of C' is glued to the left edge of C in a way that "flips" orientation.



Now I leave it to you to glue C and C' together along the edges, as indicated; you wil get a Mobius strip.

Proof of Propositions

Proof of Proposition 16.1.2. Note that in this problem, given $A \subset X$, the complement of A is the complement of A in X. That is, if A^C denotes the complement, we have

$$A^C = \{x \in X \text{ such that } x \notin A.\}.$$

1. To show the empty set is closed, we must show its complement is open. We know $\emptyset^C = X$, and by the definition of topological space, we know X is open. This shows that the empty set is closed.

To show that X is closed, we must show that its complement is open. We know $X^C = \emptyset$, and the empty set is always open (by definition of topological space). Thsi shows that X is closed.

2. Since each A_i is closed, we know A_i^C is open for every $i = 1, \ldots, n$. By DeMorgan's Laws, we have

$$(\bigcup_{i=1,\ldots,n} A_i)^C = \bigcap_{i=1,\ldots,n} (A_i^C).$$

The righthand side is a finite intersection open sets. Hence it is open (by definition of topological space). Because $(\bigcup_{i=1,\ldots,n} A_i)^C$ is open, we conclude that $\bigcup_{i=1,\ldots,n} A_i$ is closed (by definition of closed set). This finishes the proof.

3. Again by DeMorgan's Laws, we have

$$(\bigcap_{\alpha \in \mathcal{A}} A_\alpha)^C = \bigcup_{\alpha \in \mathcal{A}} (A_\alpha^C).$$

Each A_α^C is open because each A_α is closed; thus the righthand side is a union of open sets. Thus the righthand side is open (by definition of topological space). This shows that $(\bigcap_{\alpha \in \mathcal{A}} A_\alpha)^C$ is open, which means $\bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is closed (by definition of closed set). □

Proof of Proposition 16.1.3. Let $A \subset Y$ be closed. Then A^C is open. Moreover,

$$f^{-1}(A^C) = f^{-1}(A)^C.$$

(To see this, you need to exhibit each set as a subset of the other. Well, x is in the lefthand side if $f(x) \notin A$. In particular, $x \notin f^{-1}(A)$. Likewise, if x is in the righthand side, then $x \notin f^{-1}(A)$, so $f(x) \notin A$, meaning $x \in f^{-1}(A^C)$.)

And the lefthand side is open by definition of continuous map. Thus $f^{-1}(A)^C$ is open, meaning $f^{-1}(A)$ is closed. □

Proof of Proposition 16.1.4. Omitted until next time. □

Proof of Proposition 16.2.2. Let us recall the definition of the disjoint union $\coprod_{\alpha, \mathcal{A}} U_\alpha$—this is the set of all pairs (x, α) where $\alpha \in \mathcal{A}$ and $x \in U_\alpha$. Then the function p is given by

$$p : \coprod_{\alpha \in \mathcal{A}} U_\alpha \to X, \qquad (x, \alpha) \mapsto x.$$

That is, $p(x, \alpha) = x$.

Because $\{U_\alpha\}$ is a cover of X, for every $x \in X$, there is some α such that $x \in U_\alpha$. In particular, for every $x \in X$, there is some (x, α) such that $p(x) = x$. This shows that p is a surjection.

In a previous lecture, we showed that whenever p is a surjection, then the induced function

$$\text{domain of } p/\sim \to \text{ codomain of } p$$

is a bijection if $\sim$ is defined by $x \sim x' \iff p(x) = p(x')$. This is exactly the equivalence relation that we are taking, so we conclude that the induced map

$$f : (\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \to X$$

is a bijection. Note that we have now given this map a name: f.

First let's show f is continuous. In homework you showed that f is continuous if and only if p is. (A map from the quotient is continuous if and only if the its composition with the projection map is.) So let us show p is continuous. This means we must show that if $V \subset X$ is an open subset, then $p^{-1}(V)$ is open. By definition of coproduct topology, $p^{-1}(V)$ is open if and only if

$$p^{-1}(V) \cap U_\alpha$$

is open for every $\alpha \in \mathcal{A}$. So let's prove it. First, we compute:

$$\begin{aligned} p^{-1}(V) \cap U_\alpha &= \{\ x \in U_\alpha \text{ such that } p(x) \in V\} && (16.3.0.1) \\ &= \{\ x \in U_\alpha \text{ such that } x \in V && (16.3.0.2) \\ &= V \cap U_\alpha. && (16.3.0.3) \end{aligned}$$

Because $V \subset X$ is open and $U_\alpha \subset X$ is open, their (finite) intersection is open. This shows that $p^{-1}(V) \cap U_\alpha$ is open for every $\alpha \in \mathcal{A}$, and hence that $p^{-1}(V)$ is open. This shows p is continuous. Thus f is continuous.

Now we must show that the inverse map

$$g : X \to (\coprod_\alpha U_\alpha)/\sim$$

is continuous. For this, it suffices to show that if a subset V in the codomain is open, then $g^{-1}(V) = f(V)$ is open. Well, something in the codomain is open if and only if its preimage under the quotient map is open (by definition of quotient topology). Thus, a subset V of the codomain is open if and only if V is the image of some open subset

$$\tilde{V} \subset \coprod_\alpha U_\alpha.$$

Again by definition of coproduct topology, $\tilde{V}$ then has the property that $\tilde{V} \cup U_\alpha$ is open for every α. But then we have

$$f(V) = p(\tilde{V}) = p(\bigcup_\alpha V \cap U_\alpha) = \bigcup_\alpha p(V \cap U_\alpha).$$

For every α, we know $V \cap U_\alpha$ is open an open subset of X (because it is a finite intersection of open sets), so $p(V \cap U_\alpha) = V \cap U_\alpha$ is an open subset of X. In other words, the rightmost term in the above string of equalities is a union of open sets, and is hence open (by definition of topology). This shows $f(V)$ is open, which completes the proof. $\square$

Lecture 17

Open covers and subcovers

Last time you began thinking about open covers. Let me remind you:

Definition 17.0.1 (Definition 16.2.1)**.** Let $(X, \mathcal{T}_X)$ be a topological space, and fix a set $\mathcal{A}$. Fix a function $\mathcal{U} : \mathcal{A} \to \mathcal{T}_X$. For every $\alpha \in \mathcal{A}$, we will write U_α for the value of this function on α. We say that $\mathcal{U}$ is an *open cover* if

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = X.$$

Remark 17.0.2. Note that it seems I have changed the definition!

Exercise 17.0.3 (Do only if you or your group decides to.)**.** Show that this definition is equivalent to the old one: " We say that a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets of X is a *cover* if $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = X$. We further say this collection is an *open cover* if each U_α is open."

17.1 Subcovers

Definition 17.1.1. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X.[1] A collection $\{U_\beta\}_{\beta \in \mathcal{B}}$ is called a *subcover* of $\mathcal{U}$ if

1. $\bigcup_{\beta \in \mathcal{B}} U_\beta = X$, and
2. For every $\beta \in \mathcal{B}$, there exists $\alpha \in \mathcal{A}$ such that $U_\beta = U_\alpha$.

[1] Note that I am not explicitly saying that X is a topological space here; it is to be inferred from context, because I am talking about an "open" cover of X—this only makes sense if I know what the open sets of X are!

Exercise 17.1.2 (Do only if you or your group decides to.)**.** Let $\mathcal{U}$ be an open cover. Then a subcover of $\mathcal{U}$ is the same data as a choice of subset $\mathcal{B} \subset \mathcal{A}$ such that the composition

$$\mathcal{B} \to \mathcal{A} \to \mathcal{T}$$

is an open cover of X.

Prove the following:

Proposition 17.1.3. Let $\mathcal{A} = X \times \mathbb{R}_{>0}$ be the set of pairs (x, r) where $x \in X$ and r is a positive real number. Let (X, d) be a metric space, and equip it with the induced topology.

(i) The collection

$$\mathcal{U} = \{\mathrm{Ball}(x, r)\}_{(x,r)\in\mathcal{A}}$$

is an open cover of X.

(ii) Now let $\mathcal{B} \subset \mathcal{A}$ denote the set of pairs (x, r) where $x \in x$ and r is a positive *rational* number. (So $\mathcal{B} = X \times \mathbb{Q}$.) Then $\{U_\beta\}_{\beta\in\mathcal{B}}$ is a subcover of $\mathcal{U}$.

17.2 From last time

If you didn't have a chance last time, I want you to tackle the following:

17.2.1 Preimages of closed sets are closed

Proposition 17.2.1 (Proposition 16.1.3.)**.** Let $f : X \to Y$ be a continuous map of topological spaces. If $A \subset Y$ is closed, then its preimage is closed.

Conversely, suppose that $f : X \to Y$ is a function such that whenever $A \subset Y$ is closed, its preimage is closed. Then f is continuous.

17.2.2 Open covers can reconstruct the space

Proposition 17.2.2 (Proposition 16.2.2.)**.** Let $\{U_\alpha\}$ be an open cover of X. Note there is a function

$$p : \coprod_{\alpha\in\mathcal{A}} U_\alpha \to X.$$

Then the induced map
$$(\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \;\to X$$
is a homeomorphism. (Here, the equivalence relation $\sim$ is the one for which $x \sim x' \iff p(x) = p(x')$.

17.2.3 Closures

Proposition 17.2.3 (Proposition 16.1.4.)**.** Let $B \subset X$ be an arbitrary subset. There exists a subset, $\overline{B} \subset X$, satisfying the following properties:

1. $B \subset \overline{B}$
2. $\overline{B}$ is closed.
3. Moreover, if C is any other closed subset of X containing B, then C contains $\overline{B}$.

Informally, this means that $\overline{B}$ is the "smallest" closed subset of X containing B.

17.2.4 A Mobius band in $\mathbb{R}P^2$

Convince yourselves that there is a Möbius band inside the union $U_1 \cup U_2 \subset \mathbb{R}P^2$.

17.3 Summary of this week's knowledge

17.3.1 Closed subsets

I now expect you to know the following about closed subsets:

1. The definition of a closed subset (of a topological space).
2. That $\emptyset$ and X are both closed (as subsets of the topological space X).
3. That the intersection of arbitrarily many closed subsets is a closed subset.
4. That the union of finitely many closed subsets is a closed subset.

5. That $f : X \to Y$ is continuous iff the preimage of any closed subset is closed.

6. You should be able to give examples of closed subsets (especially because I expect you to give examples of open subsets!).

7. (We may not have had time to go over closures, so I don't expect you to know about them yet.)

17.3.2 Open covers and subcovers

I expect you to know the following about open covers:

1. Definition of cover.

2. Definition of open cover.

3. Definition of a subcover (of an open cover).

4. That given an open cover, you can reconstruct the space X you are covering. (You should know the construction by heart, even if you cannot prove that the construction is homeomorphic to X.)

5. You should be able to give me at least one example of an open cover and a subcover.

17.3.3 Compactness

On homework, you will also learn about compact spaces. I expect you to know the following:

1. Definition of compact topological space.

2. That if $f : X \to Y$ is continuous, then any compact subspace of X has compact image.

Lecture 18

Solutions to polynomial equations are closed

18.0.1 Open covers can reconstruct the space

If you haven't completed the proof of this proposition, I want you to keep working on it. It will give you practice with coproducts, quotients, the quotient topology, and homeomorphisms:

Proposition 18.0.1 (Proposition 16.2.2.)**.** Let $\{U_\alpha\}$ be an open cover of X. Note there is a function

$$p : \coprod_{\alpha \in \mathcal{A}} U_\alpha \to X.$$

Then the induced map

$$(\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \to X$$

is a homeomorphism. (Here, the equivalence relation $\sim$ is the one for which $x \sim x' \iff p(x) = p(x')$).

18.1 Familiar (?) examples of continuous functions

Going forward, you may rely on the following:

Exercise 18.1.1 (Do only if you want to.)**.** Show that addition,

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x_1, x_2) \mapsto x_1 + x_2$$

is continuous. (Here, $\mathbb{R}$ is given the topology induced by the standard metric.)

Exercise 18.1.2 (Do only if you want to.)**.** Show that the multiplication function

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x_1, x_2) \mapsto x_1 x_2$$

is continuous. (Here, $\mathbb{R}$ is given the topology induced by the standard metric.)

Exercise 18.1.3 (Do only if you want to.)**.** Show that the following functions are continuous:

1. Fix a real number $a \in \mathbb{R}$. The constant function

$$\mathbb{R} \to \mathbb{R}, \qquad x \mapsto a.$$

2. Fix two continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. The function

$$\mathbb{R} \to \mathbb{R} \times \mathbb{R}, \qquad x \mapsto (f(x), g(x)).$$

18.2 Polynomial functions are continuous

Exercise 18.2.1 (Do only if you want to.)**.** (You will need to rely on the exercises above. If you want, you can try proving the following propositions *without* proving the exercises yourself, but taking their truth for granted.)

1. Any polynomial function in one variable is continuous. That is, if one has a finite collection of real numbers $a_0, \ldots, a_n$, the function

$$p : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n = \sum_{i=0}^{n} a_i x^i$$

is continuous. (Hint: Induction on n.)

2. Any polynomial function in finitely many variables is continuous. That is, if we are given a real number $a_{i_1,\ldots,i_m}$ for some finite collection of m-tuples of non-negative integers $i_1, \ldots, i_m$, the function

$$\mathbb{R}^m \to \mathbb{R}, \qquad (x_1, \ldots, x_m) \mapsto \sum_{i_1,\ldots,i_m} a_{i_1,\ldots,i_m} x_1^{i_1} \ldots x_m^{i_m}$$

is continuous. (Hint: A lot of induction.)

18.3 Some closed subsets of $\mathbb{R}^n$

Prove the following:

Proposition 18.3.1. 1. Fix a real number $b \in \mathbb{R}$. Then the (singleton) set $\{b\} \subset \mathbb{R}$ is closed.

2. For every $m \geq 1$, the $(m-1)$-dimensional sphere

$$S^{m-1} \subset \mathbb{R}^m$$

is a closed subset of $\mathbb{R}^m$. (Recall that

$$S^{m-1} := \{(x_1, \ldots, x_m) \text{ such that } \sum_{i=1}^{m} x_i^2 = 1\}.$$

As a hint, you can use the fact that for continuous functions, preimages of closed subsets are closed.)

3. More generally, given any polynomial p in m variables, the set

$$\{x \text{ such that } p(x) = 0\} \subset \mathbb{R}^m$$

is a closed subset.

4. Even more generally, given a finite collection of polynomials $p_1, \ldots, p_k$ in m variables, the set

$$\{x \text{ such that } p_i(x) = 0 \text{ for all } i\} \subset \mathbb{R}^m$$

is a closed subset.

5. Even more generally, given an arbitrary collection of polynomials $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ in m variables, the set

$$\{x \text{ such that } p_\alpha(x) = 0 \text{ for every } \alpha \in \mathcal{A}\} \subset \mathbb{R}^m$$

is a closed subset.

Prove the following:

Proposition 18.3.2.

1. Fix a real number a. Then the set

$$(-\infty, a] \subset \mathbb{R}$$

is closed (under the standard topology).

2. Fix a real number a and let $p : \mathbb{R}^m \to \mathbb{R}$ be a polynomial function in m variables. Then the set

$$\{x \in \mathbb{R}^m \text{ such that } p(x) \leq a\ \}$$

is closed. If you need to, do the same for $\geq a$ rather than $\leq a$.

18.4 The Heine-Borel Theorem

If you have gotten this far, you can go onto facts that will be useful and that we will cover later.

Definition 18.4.1. A subset $A \subset \mathbb{R}$ is called *bounded* if there exists some positive real number $a \in \mathbb{R}$ for which

$$A \subset (-a, a).$$

More generally, given a subset $A \subset \mathbb{R}^n$, we say that A is bounded if there exists some positive real number $a \in \mathbb{R}$ for which

$$A \subset \mathrm{Ball}(0; a).$$

Prove:

Theorem 18.4.2 (Heine-Borel Theorem)**.** A subset $A \subset \mathbb{R}^n$ is compact if and only if it is both closed and bounded.

Lecture 19

Closed balls in metric spaces; Heine-Borel

19.1 The metric function is continuous

Proposition 19.1.1. Let $d : X \times X \to \mathbb{R}$ be a metric. Endow X with the metric topology (i.e., the topology induced by the metric) and endow $X \times X$ with the product topology. $\mathbb{R}$ has the standard topology.

1. Show that d is continuous.

2. For any $x_0 \in X$, show that the function

$$d(x_0, -) : X \to \mathbb{R}, \qquad x \mapsto d(x_0, x)$$

is continuous.

Proposition 19.1.2. Let $d : X \times X \to \mathbb{R}$ be a metric. Endow X with the metric topology (i.e., the topology induced by the metric) and endow $X \times X$ with the product topology.

1. Fix a real number $a \in \mathbb{R}$. For every $x_0 \in X$, show that

$$\{ \ x \in X \text{ such that } d(x_0, x) = a \ \}$$

is a closed subset of X.

2. Fix a real number $a \in \mathbb{R}$. For every $x_0 \in X$, show that
$$\{ x \in X \text{ such that } d(x_0, x) \leq a \}$$
is a closed subset of X. This is called the *closed ball of radius a centered at x_0*.

19.2 Proving Heine-Borel

Definition 19.2.1 (Definition 18.4.1)**.** A subset $A \subset \mathbb{R}$ is called *bounded* if there exists some positive real number $a \in \mathbb{R}$ for which
$$A \subset (-a, a).$$
More generally, given a subset $A \subset \mathbb{R}^n$, we say that A is bounded if there exists some positive real number $a \in \mathbb{R}$ for which
$$A \subset \mathrm{Ball}(0; a).$$

Prove:

Theorem 19.2.2 (Heine-Borel Theorem, 18.4.2.)**.** A subset $A \subset \mathbb{R}^n$ is compact if and only if it is both closed and bounded.

19.3 Summary of this week's knowledge

19.3.1 Examples of closed subsets, and polynomials

1. Polynomials are continuous functions
2. Solutions to polynomials are closed subsets
3. Solutions to inequalities defined by polynomials are closed subsets
4. In a metric space, closed balls are closed subsets

19.3.2 Closed and bounded subsets

1. You should know the Heine-Borel theorem, even if you don't know its proof.
2. You should be able to give examples of closed and bounded subsets of $\mathbb{R}^n$.

Lecture 20

Proving the Heine-Borel Theorem

Recall the following definitions:

Definition 20.0.1. A topological space is called *compact* if every open cover of the space admits a finite subcover.

Definition 20.0.2. A subset of a topological space X is called *closed* if its complement is open.

Definition 20.0.3. A subset A of $\mathbb{R}^n$ is called *bounded* if there is some $r \in \mathbb{R}$ such that

$$A \subset \mathrm{Ball}(0; r).$$

(That is, A is contained in a ball of radius r centered at the origin.)

Today, we will prove:

Theorem 20.0.4 (Heine-Borel theorem)**.** Fix $A \subset \mathbb{R}^n$. Then A is compact if and only if it is closed and bounded.

We will see pay-offs next class.

20.1 Just take these for granted

Here are a few lemmas. You should take them for granted; no need to prove them. Just read them and try to understand them.

Lemma 20.1.1. Fix two real numbers a, b such that $a \leq b$. Then the interval $[a, b]$ is compact. (We endow $[a, b] \subset \mathbb{R}$ with the subspace topology.)

Lemma 20.1.2. Let X and Y be compact. Then $X \times Y$ is compact.

Lemma 20.1.3 (You proved this in homework)**.** Let X and Y be topological spaces.

1. Let X be compact. Then any closed subset $A \subset X$ is compact.

2. Let Y be Hausdorff. Then any compact subset $B \subset Y$ is closed.

20.2 A proof of Heine-Borel

In your groups, read the following proof of the Heine-Borel theorem. Speak out when you do not understand some portion of the proof. Make sure you understand every step.

Proof. Fix n.

(Compact $\implies$ closed and bounded.) To begin, define a collection of open balls as follows:

$$W_r = \text{Ball}(0; r) \subset \mathbb{R}^n, \qquad r > 0.$$

Note that the collection $\{W_r\}_{r>0}$ forms an open cover of $\mathbb{R}^n$.

Now let $A \subset \mathbb{R}^n$ be compact, and define

$$U_r = W_r \cap A.$$

Then the collection $\{U_r\}$ forms an open cover of A. By compactness of A, there is a finite subcover, meaning there is a finite collection $r_1, \ldots, r_n$ such that

$$\bigcup_{i \in 1, \ldots, n} U_{r_i} = A.$$

But if $r > r'$, clearly $U_r \supset U_{r'}$, so letting $R = \max\{r_1, \ldots, r_n\}$, we have that $A \subset W_R$. This shows A is bounded.

To show A is closed, we simply cite Lemma 20.1.3(2). (Note that $\mathbb{R}^n$ is Hausdorff because it is a metric space.)

(Closed and bounded $\implies$ compact.) Now suppose $A \subset \mathbb{R}^n$ is closed and bounded. Well, because A is bounded, there is some real number r so that $A \subset \text{Ball}(0; r)$. In particular, A is contained in the square

$$[-r, r] \times \ldots \times [-r, r] \subset \mathbb{R}^n.$$

Here, the lefthand side consists of those points

$$(x_1, \ldots, x_n)$$

such that $x_i \in [-r, r]$ for all i. But the interval $[-r, r]$ is compact by Lemma 20.1.1; so by Lemma 20.1.2, the direct product

$$[-r, r] \times \ldots \times [-r, r]$$

is also compact. Moreover, we have

$$A \subset [-r, r] \times \ldots \times [-r, r] \subset \mathbb{R}^n$$

and because A is closed in $\mathbb{R}^n$, we know that A is closed in $[-r, r] \times \ldots [-r, r]$. Invoking Lemma 20.1.3(1), we conclude that A is compact. $\square$

Just to make sure you understood every element of the proof:

1. We used induction at some point. Where?

2. At some point we had to verify that the subspace topology of $[-r, r] \times \ldots \times [-r, r] \subset \mathbb{R}^n$ equals the product topology of $[-r, r] \times \ldots \times [-r, r]$. Where was the first point we needed to do this?

20.3 Another proof to verify

Remark 20.3.1. We will not give a Proof of Lemma 20.1.1; it is proven in most analysis classes.

Make sure you understand every step. This proof should make for good group discussion.

Proof of Lemma 20.1.2. (I) Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of $X \times Y$. Before we treat a more general case, let us assume that every element of $\mathcal{U}$ is of the form $V_\alpha \times W_\alpha$, where $V_\alpha \subset X$ is open and $W_\alpha \subset Y$ is open.

For every $x \in X$, let us consider the subset $\{x\} \times Y$. By definition of cover, for every element $(x, y) \in \{a\} \times Y$, there is some α such that $(x, y) \in V_\alpha \times W_\alpha$. So choose some collection $\mathcal{A}_x \subset \mathcal{A}$ so that

$$\bigcup_{\alpha \in \mathcal{A}_x} V_\alpha \times W_\alpha \supset \{x\} \times Y.$$

Then the collection $\{W_\alpha\}_{\alpha \in \mathcal{A}_x}$ is an open cover of Y. Because Y is compact, there is some finite subcover—i.e., some finite subset $F_x \subset \mathcal{A}_x$ so that

$$\bigcup_{\alpha \in F_x} W_\alpha = Y.$$

Now consider the finite intersection

$$\bigcap_{\alpha \in F_x} V_\alpha.$$

This is an open subset of X, and we call it V_x. Note that $x \in V_x$, and

$$\bigcup_{\alpha \in F_x} V_x \times W_\alpha \supset \{x\} \times Y. \tag{20.3.0.1}$$

In this way, for every $x \in X$, we obtain an open subset $V_x \subset X$ such that $x \in V_x$, and such that there exists some finite subset $F_x \subset \mathcal{A}$ for which (20.3.0.1) holds.

The collection $\{V_x\}_{x \in X}$ forms an open cover of X. By compactness of X, there exists a finite subcover. Hence there is some finite collection of points $x_1, \ldots, x_n \in X$ so that

$$V_{x_1} \cup \ldots \cup V_{x_n} = X.$$

It follows that

$$\bigcup_{x_1, \ldots, x_n} \bigcup_{\alpha \in F_{x_i}} V_{x_i} \times W_\alpha = X \times Y.$$

Note that the collection $\{(x_i, \alpha) \text{ such that } \alpha \in F_{x_i}\ \}$ is a finite set, while $V_{x_i} \subset V_\alpha$. Hence we have found a finite subcover:

$$\{V_\alpha \times W_\alpha\}_{\{(x_i, \alpha) \text{ such that } \alpha \in F_{x_i}\ \}}$$

(II) Now, for the general case. If $\mathcal{U} = \{U_\beta\}$ is an arbitrary open cover of $X \times Y$, for every β, let us choose a set $\mathcal{C}_\beta$ and open subsets of X and of Y so that

$$U_\beta = \bigcup_{\gamma \in \mathcal{C}_\beta} V_\gamma \times W_\gamma. \tag{20.3.0.2}$$

Let $\mathcal{A} = \bigcup_{\beta \in \mathcal{B}} \mathcal{C}_\beta$; then we have an open cover

$$\{V_\gamma \times W_\gamma\}_{\gamma \in \mathcal{A}}.$$

We produced a finite subcover of such a collection in (I). So let $\mathcal{A}' \subset \mathcal{A}$ be the finite subset for which

$$\{V_\gamma \times W_\gamma\}_{\gamma \in \mathcal{A}'}$$

is an open cover of $X \times Y$. For every $\gamma \in \mathcal{A}'$, there exists some $\beta(\gamma) \in \mathcal{B}$ for which

$$V_\gamma \times W_\gamma \subset U_{\beta(\gamma)}$$

by design (20.3.0.2). Thus we find

$$X \times Y \subset \bigcup_{\gamma \in \mathcal{A}'} U_\gamma \times W_\gamma \subset \bigcup_{\beta(\gamma)} U_{\beta(\gamma)} \subset X \times Y.$$

In other words, the collection

$$\{U_{\beta(\gamma)}\}$$

is a finite subcover of $\mathcal{U}$.

$\square$

Proofs of Propositions and selected exercises (Lectures 16 - 20)

Proposition 16.1.2 Let $(X, \mathcal{T})$ be a topological space. For this problem, we let $\mathcal{K}$ denote the collection of closed subsets. Show the following are true:

1. $\emptyset, X \in \mathcal{K}$.
2. If $A_1, \ldots, A_n \in \mathcal{K}$ is a finite collection, then $\bigcup_{i=1,\ldots,n} A_i$ is in $\mathcal{K}$.
3. For an arbitrary collection $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ of elements of $\mathcal{K}$, we have that the intersection $\bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is also in $\mathcal{K}$.

Proof of Proposition 16.1.2. Note that in this problem, given $A \subset X$, the complement of A is the complement of A in X. That is, if A^C denotes the complement, we have

$$A^C = \{x \in X \text{ such that } x \notin A.\}.$$

1. To show the empty set is closed, we must show its complement is open. We know $\emptyset^C = X$, and by the definition of topological space, we know X is open. This shows that the empty set is closed.

To show that X is closed, we must show that its complement is open. We know $X^C = \emptyset$, and the empty set is always open (by definition of topological space). Thsi shows that X is closed.

2. Since each A_i is closed, we know A_i^C is open for every $i = 1, \ldots, n$. By DeMorgan's Laws, we have

$$(\bigcup_{i=1,\ldots,n} A_i)^C = \bigcap_{i=1,\ldots,n} (A_i^C).$$

The righthand side is a finite intersection open sets. Hence it is open (by definition of topological space). Because $(\bigcup_{i=1,\ldots,n} A_i)^C$ is open, we conclude that $\bigcup_{i=1,\ldots,n} A_i$ is closed (by definition of closed set). This finishes the proof.

3. Again by DeMorgan's Laws, we have

$$(\bigcap_{\alpha \in \mathcal{A}} A_\alpha)^C = \bigcup_{\alpha \in \mathcal{A}} (A_\alpha^C).$$

Each A_α^C is open because each A_α is closed; thus the righthand side is a union of open sets. Thus the righthand side is open (by definition of topological space). This shows that $(\bigcap_{\alpha \in \mathcal{A}} A_\alpha)^C$ is open, which means $\bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is closed (by definition of closed set). $\square$

Proposition 16.1.3 Let $f : X \to Y$ be a continuous map of topological spaces. Show that if $A \subset Y$ is closed, then its preimage is closed.

Conversely, suppose that $f : X \to Y$ is a function such that whenever $A \subset Y$ is closed, its preimage is closed. Prove that f is continuous.

Proof of Proposition 16.1.3. Let $A \subset Y$ be closed. Then A^C is open. Moreover,

$$f^{-1}(A^C) = f^{-1}(A)^C.$$

(To see this, you need to exhibit each set as a subset of the other. Well, x is in the lefthand side if $f(x) \notin A$. In particular, $x \notin f^{-1}(A)$. Likewise, if x is in the righthand side, then $x \notin f^{-1}(A)$, so $f(x) \notin A$, meaning $x \in f^{-1}(A^C)$.)

And the lefthand side is open by definition of continuous map. Thus $f^{-1}(A)^C$ is open, meaning $f^{-1}(A)$ is closed. $\square$

Proposition 16.1.4. Let $B \subset X$ be an arbitrary subset. Show that there exists a subset, $\overline{B} \subset X$, satisfying the following properties:

1. $B \subset \overline{B}$
2. $\overline{B}$ is closed.
3. Moreover, if C is any other closed subset of X containing B, then C contains $\overline{B}$.

Proof of Proposition 16.1.4. Given B, let $\mathcal{K}_B$ denote the collection of all subsets $K \subset X$ for which (i) K is closed, and (ii) K contains B. Note that $\mathcal{K}_B$ is non-empty because it contains X. Now we define

$$\overline{B} := \bigcap_{K \in \mathcal{K}_B} K.$$

We see $\overline{B}$ is closed because arbitrary intersections of closed sets are closed (proving 2.). We also see that $\overline{B}$ contains B because B is contained in every $K \in \mathcal{K}_B$ (proving 1). Finally, if C is any other closed subset of X containing B, then $C \in \mathcal{K}_B$, so in particular, $C \supset \bigcap_{K \in \mathcal{K}_B} K$ (proving 3). $\square$

Proposition 16.2.2. Let $\{U_\alpha\}$ be an open cover of X. Note there is a function

$$p : \coprod_{\alpha \in \mathcal{A}} U_\alpha \to X.$$

Prove that the induced map

$$(\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \to X$$

is a homeomorphism. (Here, the equivalence relation $\sim$ is the one for which $x \sim x' \iff p(x) = p(x')$.

Proof of Proposition 16.2.2. Let us recall the definition of the disjoint union $\coprod_{\alpha,\mathcal{A}} U_\alpha$—this is the set of all pairs (x, α) where $\alpha \in \mathcal{A}$ and $x \in U_\alpha$. Then the function p is given by

$$p : \coprod_{\alpha \in \mathcal{A}} U_\alpha \to X, \qquad (x, \alpha) \mapsto x.$$

That is, $p(x, \alpha) = x$.

Because $\{U_\alpha\}$ is a cover of X, for every $x \in X$, there is some α such that $x \in U_\alpha$. In particular, for every $x \in X$, there is some (x, α) such that $p(x) = x$. This shows that p is a surjection.

In a previous lecture, we showed that whenever p is a surjection, then the induced function

$$\text{domain of } p/\sim \to \text{ codomain of } p$$

is a bijection if $\sim$ is defined by $x \sim x' \iff p(x) = p(x')$. This is exactly the equivalence relation that we are taking, so we conclude that the induced map

$$f : (\coprod_{\alpha \in \mathcal{A}} U_\alpha)/\sim \to X$$

is a bijection. Note that we have now given this map a name: f.

First let's show f is continuous. In homework you showed that f is continuous if and only if p is. (A map from the quotient is continuous if and only if the its composition with the projection map is.) So let us show p is continuous. This means we must show that if $V \subset X$ is an open subset, then $p^{-1}(V)$ is open. By definition of coproduct topology, $p^{-1}(V)$ is open if and only if

$$p^{-1}(V) \cap U_\alpha$$

is open for every $\alpha \in \mathcal{A}$. So let's prove it. First, we compute:

$$\begin{aligned} p^{-1}(V) \cap U_\alpha &= \{\ x \in U_\alpha \text{ such that } p(x) \in V\} \\ &= \{\ x \in U_\alpha \text{ such that } x \in V \\ &= V \cap U_\alpha. \end{aligned}$$

Because $V \subset X$ is open and $U_\alpha \subset X$ is open, their (finite) intersection is open. This shows that $p^{-1}(V) \cap U_\alpha$ is open for every $\alpha \in \mathcal{A}$, and hence that $p^{-1}(V)$ is open. This shows p is continuous. Thus f is continuous.

Now we must show that the inverse map

$$g : X \to (\coprod_\alpha U_\alpha)/\sim$$

is continuous. For this, it suffices to show that if a subset V in the codomain is open, then $g^{-1}(V) = f(V)$ is open. Well, something in the codomain is open if and only if its preimage under the quotient map is open (by definition of quotient topology). Thus, a subset V of the codomain is open if and only if V is the image of some open subset

$$\tilde{V} \subset \coprod_\alpha U_\alpha.$$

Again by definition of coproduct topology, $\tilde{V}$ then has the property that $\tilde{V} \cup U_\alpha$ is open for every α. But then we have

$$f(V) = p(\tilde{V}) = p(\bigcup_\alpha V \cap U_\alpha) = \bigcup_\alpha p(V \cap U_\alpha).$$

For every α, we know $V \cap U_\alpha$ is open an open subset of X (because it is a finite intersection of open sets), so $p(V \cap U_\alpha) = V \cap U_\alpha$ is an open subset of X. In other words, the rightmost term in the above string of equalities is a union of open sets, and is hence open (by definition of topology). This shows $f(V)$ is open, which completes the proof. $\square$

Proposition 17.1.3 Let $\mathcal{A} = X \times \mathbb{R}_{>0}$ be the set of pairs (x, r) where $x \in X$ and r is a positive real number. Let (X, d) be a metric space, and equip it with the induced topology.

(i) The collection

$$\mathcal{U} = \{\mathrm{Ball}(x, r)\}_{(x,r)\in\mathcal{A}}$$

is an open cover of X.

(ii) Now let $\mathcal{B} \subset \mathcal{A}$ denote the set of pairs (x, r) where $x \in x$ and r is a positive *rational* number. (So $\mathcal{B} = X \times \mathbb{Q}$.) Then $\{U_\beta\}_{\beta\in\mathcal{B}}$ is a subcover of $\mathcal{U}$.

Proof of 17.1.3. (i) By definition of metric topology (i.e., the topology induced by the metric), every $\mathrm{Ball}(x, r)$ is open. Moreover, for ever $x \in X$, clearly $x \in \mathrm{Ball}(x, r)$ for any $r > 0$, so we conclude $X \subset \bigcup_{(x,r)\in\mathcal{A}} \mathrm{Ball}(x, r)$. On the other hand, the union $\bigcup_{(x,r)\in\mathcal{A}}$ is clearly a subset of X, being a union of subsets of X. This proves that $\mathcal{U}$ is an open cover.

(ii) There is a typo; $\mathcal{B}$ does not equal $X \times \mathbb{Q}$, but it equals $X \times \mathbb{Q}_{>0}$—i.e., X times the set of positive rational numbers. Regardless, for any rational positive number r and any $x \in X$, we have that $x \in \mathrm{Ball}(x, r)$, so we again have that $X = \bigcup_{(x,r)\in\mathcal{B}} \mathrm{Ball}(x, r)$. This proves the claim. $\square$

Exercise 17.0.3 Show that this definition is equivalent to the old one: " We say that a collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of subsets of X is a *cover* if $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = X$. We further say this collection is an *open cover* if each U_α is open."

Solution to Exercise 17.0.3. The notation $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ means that we have some set $\mathcal{A}$, and for every $\alpha \in \mathcal{A}$, we have specified some open subset $U_\alpha \subset X$. That is the same information as giving a function from $\mathcal{A}$ to $\mathcal{T}$. And of course, if $U_\alpha \in \mathcal{T}$, it is open by definition.

The "cover" part of the definitions are identical, so there is nothing to check there. $\square$

Exercise 17.1.2 Let $\mathcal{U}$ be an open cover. Then a subcover of $\mathcal{U}$ is the same data as a choice of subset $\mathcal{B} \subset \mathcal{A}$ such that the composition

$$\mathcal{B} \to \mathcal{A} \to \mathcal{T}$$

is an open cover of X.

Solution to Exercise 17.1.2. As stated, the exercise isn't quite correct; we'll see why. Suppose you have an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$.

The first definition (17.1.1) says $\{U_\beta\}_{\beta \in \mathcal{B}}$ is a subcover if (i) if the union $\bigcup_{\beta \in \mathcal{B}} U_\beta$ is equal to X, (ii) for every β, there is an α so that $U_\alpha = U_\beta$.

The second definition (17.1.2) is identical for (i). Above, (ii) says we can find a function $i : \mathcal{B} \to \mathcal{A}$ so that $U_{i(\beta)} = U_\alpha$.

□

Exercise 18.1.1 Show that addition,

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x_1, x_2) \mapsto x_1 + x_2$$

is continuous. (Here, $\mathbb{R}$ is given the topology induced by the standard metric.)

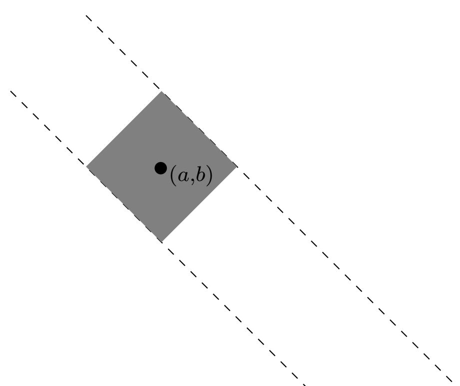
Proof of 18.1.1. For notation's sake, let's call the addition function f, so that $f(x_1, x_2) = x_1 + x_2$. We will use the ϵ-δ criterion to prove that f is continuous.

Fix $x = (a, b) \in \mathbb{R}^2$ and fix $\epsilon > 0$. Then

$$U := f^{-1}((a + b - \epsilon, a + b + \epsilon))$$

is the region in $\mathbb{R}^2$ contained (strictly) between the two lines $x_1 + x_2 = a + b - \epsilon$ and $x_1 + x_2 = a + b + \epsilon$. We must now find δ so that the open ball of radius δ around (a, b) is contained in U.

For this let us use some geometry. Clearly, the open diamond/rhombus of total width 2ϵ and total height 2ϵ, centered at (a, b), is contained in U.



In turn, the open ball of radius $\sqrt{\epsilon/2}$ is contained in this open rhombus. Thus setting $\delta = \sqrt{\epsilon/2}$, we are finished. □

Exercise 18.1.2 Show that the multiplication function

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x_1, x_2) \mapsto x_1 x_2$$

is continuous. (Here, $\mathbb{R}$ is given the topology induced by the standard metric.)

Proof of 18.1.2. Fix a point $(a, b) \in \mathbb{R}^2$. The note that for any $d \in \mathbb{R}$, we have that

$$(a + d)(b + d) = ab + (b + a)d + d^2.$$

And in particular,

$$d_{\mathbb{R}_{std}}(ab, (a + d)(b + d)) = |(b + a)d + d^2| \leq |b + a||d| + |d|^2.$$

Note that given $\epsilon > 0$, the sum $|b + a||d| + |d|^2$ is less than ϵ if each term of the sum is less than $\epsilon/2$—that is, if

$$|b + a||d| < \epsilon/2 \qquad \text{and} \qquad |d|^2 < \epsilon/2.$$

So let δ be any positive real number such that

$$\delta < \min\{\epsilon/2(|b + a|), \sqrt{\epsilon/2}\}.$$

Then we are finished. $\square$

Exercise 18.1.3 Show that the following functions are continuous:

1. Fix a real number $a \in \mathbb{R}$. The constant function

$$\mathbb{R} \to \mathbb{R}, \qquad x \mapsto a.$$

2. Fix two continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. The function

$$\mathbb{R} \to \mathbb{R} \times \mathbb{R}, \qquad x \mapsto (f(x), g(x)).$$

Proof of 18.1.3. 1. Given any ϵ, any δ will do.

2. You've shown this in your homework for metric spaces. More generally, let W, X, Y be topological spaces, and fix two continuous function $f : W \to X$ and $g : W \to Y$. We will show that $h : W \to X \times Y, h(w) := ((f(w), g(w)))$, is continuous.

Let $A \subset X \times Y$ be open. By definition (of product topology),

$$A = \bigcup_{\alpha \in \mathcal{A}} U_\alpha \times V_\alpha$$

for some set $\mathcal{A}$, and where $U_\alpha \subset X$ and $V_\alpha \subset Y$ are open. Note that

$$h^{-1}(U_\alpha \times V_\alpha) = f^{-1}(U_\alpha) \cap g^{-1}(V_\alpha).$$

Because f and g are continuous, we see that $h^{-1}(U_\alpha \times V_\alpha)$ is thus an intersection of two open sets—thus, $h^{-1}(U_\alpha \times V_\alpha)$ is open. We conclude that

$$h^{-1}(W) = \bigcup_{\alpha \in \mathcal{A}} h^{-1}(U_\alpha \times V_\alpha)$$

so $h^{-1}(W)$ is an open subset of X (being a union of open subsets). This concludes the proof. $\square$

Exercise 18.2.1 (You will need to rely on the exercises above. If you want, you can try proving the following propositions *without* proving the exercises yourself, but taking their truth for granted.)

1. Any polynomial function in one variable is continuous. That is, if one has a finite collection of real numbers $a_0, \ldots, a_n$, the function

$$p : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n = \sum_{i=0}^{n} a_i x^i$$

   is continuous. (Hint: Induction on n.)

2. Any polynomial function in finitely many variables is continuous. That is, if we are given a real number $a_{i_1,\ldots,i_m}$ for some finite collection of m-tuples of non-negative integers $i_1, \ldots, i_m$, the function

$$\mathbb{R}^m \to \mathbb{R}, \qquad (x_1, \ldots, x_m) \mapsto \sum_{i_1,\ldots,i_m} a_{i_1,\ldots,i_m} x_1^{i_1} \ldots x_m^{i_m}$$

   is continuous. (Hint: A lot of induction.)

Proof of 18.2.1. 1. First, let us prove that the function $f_n : x \mapsto x^n$ is continuous. We will perform induction on the degree n. For $n = 1$ this is obvious. For $n = 2$, we note that $f_n(x) = f_1(x) \cdot f_{n-1}(x)$. This is the composition

$$\mathbb{R} \xrightarrow{(f_1, f_{n-1})} \mathbb{R} \times \mathbb{R} \xrightarrow{\text{multiplication}} \mathbb{R}.$$

The second arrow is continuous by Exercise 18.1.2. the first arrow is continuous by Exercise 18.1.32 and by induction. Because the composition of continuous functions is continuous, we conclude that f_n is continuous given that f_{n-1} is continuous.

Second, let us now note that the function $x \mapsto ax^n$ (for any constant $a \in \mathbb{R}$) is continuous. This function can again be written as a composition

$$\mathbb{R} \xrightarrow{(a, f_{n-1})} \mathbb{R} \times \mathbb{R} \xrightarrow{\text{multiplication}} \mathbb{R}.$$

which is continuous by combining the inductive proof above with Exercise 18.1.31.

Finally, we must prove that the polynomial function p is continuous. We proceed by induction by the degree n of p. For $n = 0$, p is the constant function $x \mapsto a_0$. this is continuous by a previous exercise (18.1.3 1). Now suppose that any polynomial q of degree $n - 1$ is continuous. Then p can be written as a composition

$$\mathbb{R} \xrightarrow{(q, a_n x^n)} \mathbb{R} \times \mathbb{R} \xrightarrow{\text{addition}} \mathbb{R}$$

where $q(x) = a_0 + a_1 x^1 + \dots a_{n-1} x^{n-1}$. Each function in this composition is continuous, hence so is the composition. This completes the proof of 1.

2. Omitted. $\square$

Proposition 18.3.1

1. Fix a real number $b \in \mathbb{R}$. Then the (singleton) set $\{b\} \subset \mathbb{R}$ is closed.

2. For every $m \geq 1$, the $(m-1)$-dimensional sphere

$$S^{m-1} \subset \mathbb{R}^m$$

is a closed subset of $\mathbb{R}^m$. (Recall that

$$S^{m-1} := \{(x_1, \dots, x_m) \text{ such that } \sum_{i=1}^{m} x_i^2 = 1\}.$$

As a hint, you can use the fact that for continuous functions, preimages of closed subsets are closed.)

3. More generally, given any polynomial p in m variables, the set

$$\{x \text{ such that } p(x) = 0\} \subset \mathbb{R}^m$$

is a closed subset.

4. Even more generally, given a finite collection of polynomials $p_1, \ldots, p_k$ in m variables, the set

$$\{x \text{ such that } p_i(x) = 0 \text{ for all } i\} \subset \mathbb{R}^m$$

is a closed subset.

5. Even more generally, given an arbitrary collection of polynomials $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ in m variables, the set

$$\{x \text{ such that } p_\alpha(x) = 0 \text{ for every } \alpha \in \mathcal{A}\} \subset \mathbb{R}^m$$

is a closed subset.

Proof of 18.3.1. 1. The complement $U = \mathbb{R} \setminus \{b\}$ is open. (For example, for any $x \in U$, the open ball $\mathrm{Ball}(x; |b-x|)$ is contained in U.) This shows that $\{b\} \subset \mathbb{R}$ is closed.

2. Let $p(x_1, \ldots, x_m) = x_1^2 + \ldots x_m^2$. This is a function $p : \mathbb{R}^m \to \mathbb{R}$, and is continuous because it is polynomial. Hence preimages of closed subsets are closed. Now we note that $\{1\} \subset RR$ is closed by the previous part of this problem, and we note that $p^{-1}(\{1\}) = S^{m-1}$.

3. Same proof, but by taking $\{b\} = \{0\} \subset \mathbb{R}$.

4. Given part 3., note that the set in questin is the intersection of $p_i^{-1}(\{0\})$; i.e., an intersection of closed subsets of $\mathbb{R}^m$. Hence it is closed.

4. Same proof. $\square$

Proposition 18.3.2

1. Fix a real number a. Then the set

$$(-\infty, a] \subset \mathbb{R}$$

is closed (under the standard topology).

2. Fix a real number a and let $p : \mathbb{R}^m \to \mathbb{R}$ be a polynomial function in m variables. Then the set

$$\{x \in \mathbb{R}^m \text{ such that } p(x) \leq a \}$$

is closed. If you need to, do the same for $\geq a$ rather than $\leq a$.

Proof of Proposition 19.1.1. 1. The set $U = (a, \infty) \subset \mathbb{R}$ is open. For example, for any $x \in U$, we have that the open ball $\text{Ball}(x; |a - x|)$ is contained in U. This shows $U^C = (-\infty, a]$ is closed.

2. The indicated set is $p^{-1}((-\infty, a])$. Because p is continuous (Exercise 18.2.11), and preimages of closed sets are closed sets for continuous maps, the claim follows from the previous part of this problem. $\square$

Proposition 19.1.1. Let $d : X \times X \to \mathbb{R}$ be a metric. Endow X with the metric topology (i.e., the topology induced by the metric) and endow $X \times X$ with the product topology. $\mathbb{R}$ has the standard topology.

1. Show that d is continuous.

2. For any $x_0 \in X$, show that the function

$$d(x_0, -) : X \to \mathbb{R}, \qquad x \mapsto d(x_0, x)$$

is continuous.

Proof of 19.1.1. 1. We use the ϵ-δ criterion, remembering that the product metric is given by

$$d_{X \times Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

(In this problem, Y happens to equal X.) So fix $(x_1, x_2) \in X \times X$ along with $\epsilon > 0$. For any δ, we have that

$$d_{X \times X}((x_1, x_2), (x_1', x_2')) < \delta \implies d(x_1, x_1') + d(x_2, x_2') < \delta. \tag{19.3.0.3}$$

Keep the above in mind. Now let's repeatedly apply the triangle inequality:

$$d(x_1', x_2') \leq d(x_1', x_1) + d(x_1, x_2') \leq d(x_1', x_1) + d(x_1, x_2) + d(x_2, x_2'). \tag{19.3.0.4}$$

By symmetry, we also conclude

$$d(x_1, x_2) \leq d(x_1', x_1) + d(x_1', x_2') + d(x_2, x_2'). \tag{19.3.0.5}$$

Combining (19.3.0.4) and (19.3.0.5) we obtain:

$$|d(x_1, x_2) - d(x_1', x_2')| \leq d(x_1', x_1) + d(x_2, x_2').$$

By the previous equation (19.3.0.3), we conclude

$$|d(x_1, x_2) - d(x_1', x_2')| \leq 2\delta.$$

Thus choosing δ to be any number less than $\epsilon/2$, we are finished.

2. We note that the function in question is a composition

$$X \to X \times X \xrightarrow{d} \mathbb{R}$$

where the first function sends $x \mapsto (x_0, x)$. So it suffices to prove that for any $x_0 \in X$, the "horizontal inclusion" function

$$X \to X \times, \qquad x \mapsto (x_0, x)$$

is continuous. Because X is a metric space, let us use the ϵ-δ criterion. Given ϵ, let δ be any positive number less than ϵ. Then if $d(x, x') < \epsilon$, we see that

$$d_{X \times X}((x_0, x), (x_0, x')) = d(x_0, x_0) + d(x, x') = 0 + \delta < \epsilon.$$

$\square$

Proposition 19.1.2 Let $d : X \times X \to \mathbb{R}$ be a metric. Endow X with the metric topology (i.e., the topology induced by the metric) and endow $X \times X$ with the product topology.

1. Fix a real number $a \in \mathbb{R}$. For every $x_0 \in X$, show that

   $$\{\ x \in X \text{ such that } d(x_0, x) = a\ \}$$

   is a closed subset of X.

2. Fix a real number $a \in \mathbb{R}$. For every $x_0 \in X$, show that

   $$\{\ x \in X \text{ such that } d(x_0, x) \leq a\ \}$$

   is a closed subset of X. This is called the *closed ball of radius a centered at x_0*.

Proof of 19.1.2. 1. By Proposition 18.3.1, the set $\{a\} \subset \mathbb{R}$ is closed. We know that for all $x_0 \in X$, the function $x \mapsto d(x_0, x)$ is continuous (Proposition 19.1.1(2)). Thus the preimage of $\{a\}$ is closed, and the set in question is precisely said preimage.

2. Same exact proof, except we take our closed set in $\mathbb{R}$ to be $(-\infty, a] \subset \mathbb{R}$. (This is closed by Proposition 18.3.2(1).) $\square$

Lecture 21

Connectedness

Today is a lecture day. You can drop your pens and pencils.

21.1 From last lecture

21.1.1 On writing proof

(Antoni) Gaudi is the architect who designed the Sagrada Familia in Barcelona, Spain. I paraphrased him in class, saying

> There should be light. But not too much.[1]

(Gaudi was presumably talking about the design of spaces, and how much light a space should have.)

The same applies to proofs. "Light" here is a euphemism for detail. So when I write you a proof, many details may be excluded, so as not to blind you; but the details included should give you sight of what is going on.

But there are two parts to the light advice: Even before worrying about having too much light, you should have light. Many of the proofs I've read in homework lack lighting. So blind me. Overburden me with your light.

[1] The actual quote is "The amount of light should be just right, not too much, not too little, since having too much or too little light can both cause blindness."

21.1.2 Openness and closedness

When we say a subset A is closed or open, it *matters* to specify what space A is a subset of. For example, let us consider the space

$$X = [a, b) \subset \mathbb{R}$$

and endow X with the subspace topology. Let $U = [a, a + \epsilon)$ for some small positive number ϵ (so that in particular, $U \subset X$). Then

U is an open subset of X (you should check this using the definition of the topology of X).

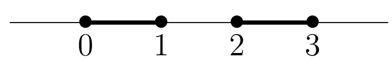
But

U is not an open subset of $\mathbb{R}$ (you should check this also).

21.2 Path-connectedness

We begin with an example.

Example 21.2.1. Let $X = [0, 1] \coprod [2, 3] \subset \mathbb{R}$, drawn below:



Would you call X connected?

Remark 21.2.2 (Properties of spaces vs. properties of subsets)**.** Above, I used that X was a subset of $\mathbb{R}$ to define the topology of X, but once we know about X's topology, we could ask the connectedness question of X (without reference to $\mathbb{R}$). Is the following space connected?

(Importantly, the picture makes no reference to $\mathbb{R}$ itself.) So unlike "closed" or "open," the adjective "connected" makes sense as a property of a space X. And when we ask whether a subset is connected, we are asking about the property of that subset as a space (endowed with the subspace topology). Aside from specifying the topology of the subspace, the parent set is irrelevant to the question of connectedness.

I want to talk today about two different ways to talk about the *connectedness* of a topological space.

This is the most intuitive definition. First, some preliminaries: We let

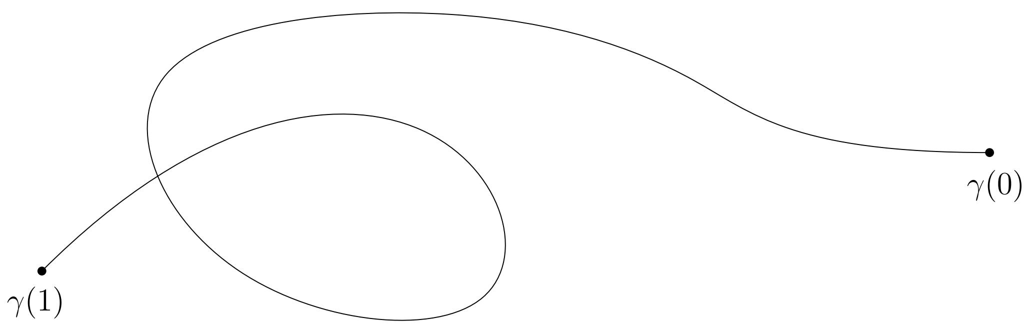
$$[0, 1]$$

denote the usual closed interval from 0 to 1. We treat it as a topological space by giving it the subspace topology inherited from $\mathbb{R}$.

Definition 21.2.3. Let X be a topological space. A *path* in X is a continuous function

$$\gamma : [0, 1] \to X.$$

Example 21.2.4. Below is an image of a possible path $\gamma : [0, 1] \to \mathbb{R}^2$.



Note that a path need not be injective (it can cross over itself).

Definition 21.2.5. Let X be a topological space, and fix a path $\gamma : [0, 1] \to X$. We say that γ is a path *from* $\gamma(0)$ *to* $\gamma(1)$.

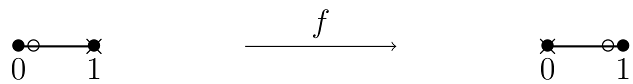
Proposition 21.2.6. Let X be a topological space, and fix $x, x' \in X$. If there exists a path from x to x', then there exists a path from x' to x.

This should be an intuitive proposition: If there's a path from x to x', you can just "reverse" the path to get from x' to x. That's the intuition we'll follow in the proof.

Proof. Consider the function

$$f : [0, 1] \to [0, 1], \qquad t \mapsto 1 - t.$$

(So for example, $f(0) = 1$ and $f(1) = 0$.)



You can check that f is continuous.

Now, let

$$\gamma : [0,1] \to X$$

be a path from x to x' (so $\gamma(0) = x$, and $\gamma(1) = x'$). Let us define

$$\overline{\gamma} = \gamma \circ f.$$

Because f and γ are continuous, the composition $\overline{\gamma}$ is. Moreover,

$$\overline{\gamma}(0) = \gamma(f(0)) = \gamma(1) = x'$$

and likewise, $\overline{\gamma}(1) = x$. Thus $\overline{\gamma}$ is a path from x' to x. $\square$

Remark 21.2.7. Let X be a topological space. Then for any $x \in X$, there exists a path from x to itself. To see this, note that the constant path

$$\gamma : [0,1] \to X, \qquad \gamma(t) = x \forall t \in [0,1]$$

is a path from x to itself.

The previous proposition says that if there is a path from x to x', then there is a path from x' to x.

Moreover, it turns out you can prove that if there is a path from x to x', and if. there is a path from x' to x'', then there is a path from x to x''. To see this, suppose we have two paths

$$\gamma' : [0,1] \to X, \gamma'' : [0,1] \to X$$

such that $\gamma'(0) = x$, $\gamma'(1) = \gamma''(0) = x'$, and $\gamma''(1) = x''$. Define a path as follows:

$$\gamma : [0,1] \to X, \qquad \gamma(t) = \begin{cases} \gamma'(2t) & t \in [0,1/2] \\ \gamma''(2t-1) & t \in [1,2,1]. \end{cases}$$

It would take us a little bit afield to prove that γ is continuous, but I promise you can prove it with the tools at your disposal. Note that

$$\gamma(0) = \gamma'(2 \cdot 0) = \gamma'(0) = x, \qquad \gamma(1) = \gamma''(2-1) = \gamma''(1) = x''$$

so γ is indeed a path from x to x''.

All this is to say that there is an equivalence relation on any topological space X given as follows: We say $x \sim x'$ if and only if there exists a path from x to x'. Though we may not see this too often in this class, there is a name for the set of equivalence classes for this relation:

$$\pi_0(X) = X/\sim.$$

The left-hand side is read "pie nought of X." It is also called the set of "path-connected components" of X.

Definition 21.2.8. Let X be a topological space. We say that X is *path-connected* if for any two points $x, x' \in X$, there exists a path from x to x'.

Example 21.2.9. Let $X = \mathbb{R}$. Then X is path-connected. To see this, fix any two points $x, x' \in X$. Then define a function γ by "drawing a straight path from x to x'." The previous sentence was vague, so let's make it precise: Define

$$\gamma : [0,1] \to X, \qquad \gamma(t) = x + t(x' - x).$$

Note that x and x' are constants (we've fixed them!) while t is the variable.

γ is a continuous function. Let's shed some light on why: Because we've given $[0,1]$ the subspace topology, the inclusion

$$[0,1] \to \mathbb{R}, \qquad t \mapsto t$$

is a continuous function. Now let $f : \mathbb{R} \to \mathbb{R}$ be the linear function $t \mapsto x + t(x' - x)$. This is continuous (for example, by previous lectures). Hence the composition

$$[0,1] \to \mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous. On the other hand, this composition is precisely γ.

Finally, note that $\gamma(0) = x$ and $\gamma(1) = x'$.

Example 21.2.10. More generally, let $X = \mathbb{R}^n$. Then X is path-connected. To see this, given x and x' in X, again define

$$\gamma : [0,1] \to \mathbb{R}^n, \qquad t \mapsto x + t(x' - x).$$

Note now that we are using vector scaling and vector addition/subtraction to define γ. I'll leave it to you to check that γ is continuous, and that γ is a path from x to x'.

By definition, the notion of path-connectedness depends on the topology of $[0,1]$ (because we need to know which functions out of $[0,1]$ are continuous). So let's see something basic about the topology of $[0,1]$:

Proposition 21.2.11. Suppose that $A \subset [0,1]$ is a subset which is both closed and open. Then A is either empty, or equal to $[0,1]$.

For this, we'll use a Lemma:

Lemma 21.2.12. If $B \subset [0,1]$ is open, and if $b \in B$ does not equal 0 or 1, then there exists some $\epsilon > 0$ so that $(b-\epsilon, b+\epsilon) \subset B$.

Proof of Lemma 21.2.12. Since $B \subset [0,1]$ is open, by definition of subspace topology, there exists $W \subset \mathbb{R}$ open so that $B = W \cap [0,1]$. Now consider the intersection $W \cap (0,1)$. This is an open subset of $\mathbb{R}$, being the intersection of two open subsets—in particular, for any $b \in W \cap (0,1)$, there exists an open ball fully contained in $W \cap (0,1)$ containing b. Let ϵ be the radius of this open ball. Then

$$(b-\epsilon, b+\epsilon) = \mathrm{Ball}(b;\epsilon) \subset W \cap (0,1) \subset W \cap [0,1] = B.$$

□

Proof of Proposition 21.2.11. First, let us recall a fact from real analysis:

> If $B \subset \mathbb{R}$ is a closed, non-empty, and bounded subset, then B has a minimal element. That is, there exists $b_0 \in B$ such that $b \in B \implies b_0 \leq b$.[2] Likewise, B has a maximal element.

We proceed by contradiction.

Let $B \subset A$ be a closed and open subset; by way of contradiction, we may assume neither B nor B^C are empty. So let us assume $0 \in B$ without loss of generality. (If $0 \notin B$, just swap the roles of B and B^C.)

Let $b_0 = \min B^C$. (Note that B^C is closed and bounded, so it has a minimum by the above fact.) Note also that $b_0 \neq 0$. Moreover, $b_1 \neq 1$—for if so, then $B^C = \{1\} \subset [0,1]$, and B^C is not an open subset of $[0,1]$.

Thus, we may use Lemma 21.2.12 to conclude that B^C must contain some interval $(b_0-\epsilon, b_0+\epsilon)$. This contradicts the minimality of $b_0 \in B^C$.

Thus, it must be that either B or B^C are empty. This completes the proof. □

[2] By the completeness axiom of the real line, take b_0 to be the infimum of the bounded set B. Then b_0 is a limit point; but B is closed, so $b_0 \in B$.

This proposition is powerful. For example, we have the following:

Corollary 21.2.13. Let X be a discrete topological space and fix elements $x, x' \in X$. Then there exists a path from x to x' if and only if $x = x'$.

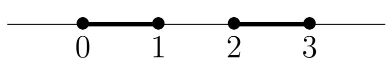
Proof. Suppose $\gamma : [0,1] \to X$ is continuous, and that x is in the image of γ. because X has the discrete topology, the singleton set $\{x\}$ is both closed and open. (To see this, recall that every subset of X is open in the discrete topology. In particular, both $\{x\}$ and its complement are open.) Thus, the preimage $\gamma^{-1}(\{x\})$ is both a closed and open subset of $[0,1]$. By Lemma 21.2.11, the preimage must be either empty or all of $[0,1]$. Because we assumed x to be in the image,

$$\gamma^{-1}(\{x\}) = [0,1].$$

In particular, γ is a constant function, so $\gamma(0) = \gamma(1) = x$. □

Example 21.2.14. So, if X is a discrete topological space with two or more elements, X is not path-connected.

Example 21.2.15. Let $X = [0,1] \coprod [2,3] \subset \mathbb{R}$, drawn below as before:



Then X is not path-connected.

Indeed, I'll take x to be some point in $[0,1]$ and x' to be some point in $[2,3]$. Suppose (for the purpose of contradiction) that there is a path

$$\gamma : [0,1] \to X$$

from x to x'. Then the composition

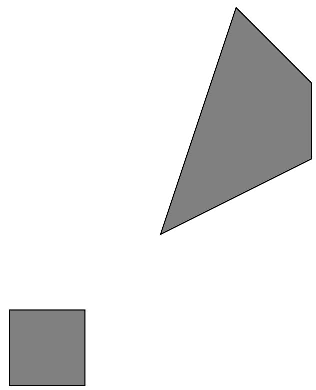
$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}$$

(where the second map is the inclusion map) is continuous. By the intermediate value theorem from calculus, for any value y such that $x \leq y \leq x'$, there must be some $t \in [0,1]$ such that $f(t) = y$.

But γ has image contained in X, and in particular, the composition f has no image in the open interval $(1,2)$. In particular, we have been led to a contradiction.

Remark 21.2.16. Note that we have used many results from your analysis class. This is because of the central role of the real line in these discussions, and because your analysis class is devoted to the study of the real line.

Example 21.2.17. Let X be the subset of $\mathbb{R}^2$ drawn below, given the subspace topology:

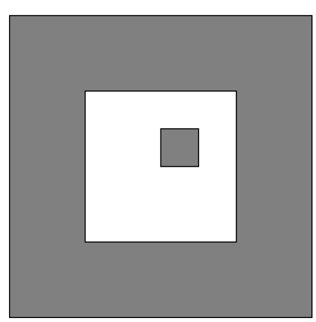


Then X is not path-connected. The proof is similar as the previous example, so I will be brief: By way of contradiction, suppose $\gamma : [0,1] \to X$ is a continuous path from x to x', where x is in the lower-right component of X and x' is in the upper-left component. Then consider the composition

$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}^2 \to \mathbb{R}$$

where the middle arrow is the inclusion, and the last arrow is the projection map sending $(x_1, x_2) \mapsto x_1$. Then f is continuous, being a composition of continuous functions; but again, f will violate the intermediate value theorem.

Example 21.2.18. Let X be the subset of $\mathbb{R}^2$ shaded below, given the subspace topology:



Then X is not path-connected. The proof is similar as the previous example, so I will be brief: By way of contradiction, suppose $\gamma : [0,1] \to X$ is a continuous path from x to x', where x is in the middle component of X and x' is in the outer component. Then consider the composition

$$f : [0,1] \xrightarrow{\gamma} X \to \mathbb{R}^2 \to \mathbb{R}$$

where the middle arrow is the inclusion, and the last arrow is now the map sending an element $y \in \mathbb{R}^2$ to the number $d(x,y)$. Then f again violates the intermediate value theorem.

21.3 Connectedness

So, path-connectedness was an intuitive notion: We'll say a space is path-connected if any two points can be connected by a path. Confusingly, the term "path-connected" is not the same as the term "connected" in our culture.

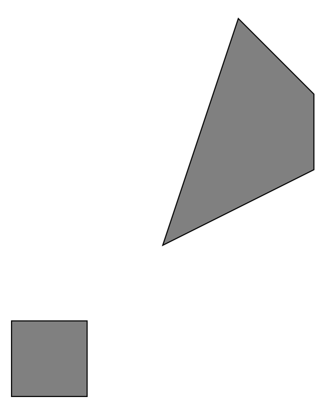
We now discuss a far less intuitive notion:

Definition 21.3.1. We say that a space X is *connected* if the following holds: If $A \subset X$ is both open and closed, then either $A = X$ or $A = \emptyset$.

Example 21.3.2. By Proposition 21.2.11, we know that $X = [0,1]$ is a connected space.

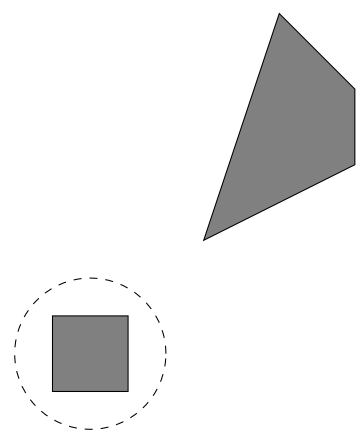
Example 21.3.3. Let X be a discrete topological space. If X has two or more elements, X is not connected.

Example 21.3.4. Let X be the subset of $\mathbb{R}^2$ drawn below, given the subspace topology:

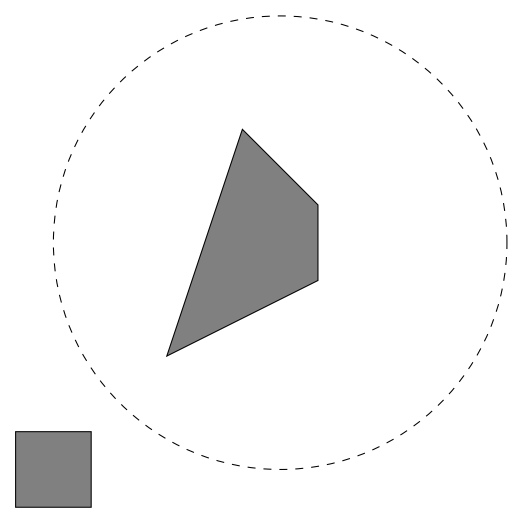


Let us label the lower-left component by A, and the upper-right component by B. I claim that both A and B are each both open and closed.

To see that A is open, simply observe that there is an open ball $W \subset \mathbb{R}^2$ for which $W \cap X = A$ (and then cite the definition of the subspace topology, which defines the topology on $X \subset RR^2$):



Because $B = A^C \subset X$, we conclude B is closed. To see B is open, likewise observe an open ball in X containing B but not A:



So B is open, meaning $A = B^C$ is closed. This shows $A \subset X$ is both open and closed, but $A \neq X$ and $A \neq \emptyset$.

Notice that all our examples connectedness/path-connectedness are the same. This is because of the following:

Proposition 21.3.5. If X is path-connected, then X is connected.

Proof. We will prove the contrapositive—that is, if X is not connected, then X is not path-connected.

Because X is not connected, there exists a subset $A \subset X$ which is non-empty, not all of X, but both open and closed.

So choose $x \in A$, and choose $x' \in A^C \subset X$. I claim there is no path from x to x'.

To see this, suppose we have a continuous map $\gamma : [0,1] \to X$ for which γ intersects A, we must have that $\gamma^{-1}(A)$ is non-empty. On the other hand, A is both open and closed, so $\gamma^{-1}(A)$ is both open and closed—this means $\gamma^{-1}(A) = [0,1]$ by Proposition 21.2.11.

That is, if $\gamma(t) \in A$ for some t, then $\gamma(t) \in A$ for every $t \in [0,1]$. In particular, if $x = \gamma(0)$, then $x' \neq \gamma(1)$. This proves the claim, and hence the proposition. □

Warning 21.3.6. There exist connected spaces that are not path-connected.

Lecture 22

More on connectedness

22.1 Some basics

Let's make explicit the following:

Proposition 22.1.1 (Inclusions are continuous.)**.** Let $A \subset X$ and give A the subspace topology. Then the inclusion function $\iota : A \to X$ given by $\iota(a) = a$ is continuous.

Proof. Suppose $W \subset X$ is open. Then $\iota^{-1}(W) = A \cap W$. This is open by definition of subspace topology. $\square$

Proposition 22.1.2 (Maps to images of continuous maps are continuous)**.** Let $f : X \to Y$ be continuous, and endow $f(X) \subset Y$ with the subspace topology. Then the function $X \to f(X)$ sending $x \mapsto f(x)$ is continuous.

Proof. Suppose $V \subset f(X)$ is open. Then there exists some subset $W \subset Y$ for which $V = W \cap f(X)$. In particular, $f^{-1}(V) = f^{-1}(W)$. The latter is open in X by definition of continuity, so $f^{-1}(V) \subset X$ is open. $\square$

Proposition 22.1.3 (Subspace topologies factor)**.** Let $X \subset Y \subset Z$ and let Z be a topological space. Then the following topologies on X are equal:

- The subspace topology $\mathcal{T}_{X \subset Z}$ of X as a subset of Z.
- The subspace topology $\mathcal{T}_{X \subset Y}$ of X as a subset of Y (where Y is given the subspace topology, induced by virtue of Y being a subset of Z).

Proof. ($\mathcal{T}_{X\subset Z} \subset \mathcal{T}_{X\subset Y}$). Let $U \in \mathcal{T}_{X\subset Z}$. Then by definition, there exists some $W \subset Z$ open for which $U = X \cap W$.

then we see that $U = X \cap W = X \cap (Y \cap W)$, where the last equality is true because $X \subset Y$. By definition of subspace topology (for Y), we see that $V = Y \cap W$ is an open subset of Y. Then $U = X \cap V$ implies that $U \in \mathcal{T}_{X\subset Y}$.

($\mathcal{T}_{X\subset Y} \subset \mathcal{T}_{X\subset Z}$). If $U \in \mathcal{T}_{X\subset Y}$, there is some open subset $V \in \mathcal{T}_Y$ for which $U = V \cap X$. By definition of subspace topology (for Y), we know there exists some $W \subset Z$ open so that $W \cap Y = V$. Hence

$$U = X \cap V = X \cap (W \cap Y) = W \cap (X \cap Y) = W \cap X$$

meaning $U \in \mathcal{T}_{X\subset Z}$.

This finishes the proof. □

22.2 An application of connectedness

Let's recall some ideas from last time. We saw two very different-looking notions of connectedness:

Definition 22.2.1 (Path-connected)**.** Let X be a topological space. We say X is path-connected if for every $x, x' \in X$, there exists a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = x'$.

Definition 22.2.2 (Connected)**.** Let X be a topological space. We say X is connected if the following hold: If $A \subset X$ is both open and closed, then A is either $\emptyset$ or X.

We also saw:

Proposition 22.2.3. The interval $[0,1]$ is connected.

Let's see one application of the idea of connectedness. The intuition for the following is that if a function f is continuous, it does not tear apart things that are connected.

Proposition 22.2.4 (Continuous functions preserve connectedness)**.** Let $f : X \to Y$ be a continuous function. If X is connected, then $f(X)$ is connected. If X is path-connected, then so is $f(X)$.

(Note that $f(X) \subset Y$ is being given the subspace topology.)

Proof of Proposition 22.2.4. Let $A \subset f(X)$ be both open and closed. Then $f^{-1}(A) \subset X$ is both open and closed. (This is because the map $X \to f(X)$ is closed by the Lemma.) Hence $f^{-1}(A)$ must either be X or $\emptyset$. The former means that A must equal $f(X)$. The latter means that A must be empty by definition of the image $f(X)$.

Hence $f(X)$ is connected.

As for path-connectedness: Let $y, y' \in f(X)$. Choose $x, x' \in X$ so that $f(x) = y$ and $f(x') = y'$. Because X is path-connected, there exists $\gamma : [0,1] \to X$ for which $\gamma(0) = x$ and $\gamma(1) = x'$. Now let $\gamma' = f \circ \gamma$. This is continuous because f and γ are. Moreover,

$$\gamma'(0) = f(\gamma(0)) = y, \qquad \gamma'(1) = f(\gamma(1)) = y'.$$

□

Exercise 22.2.5. Prove the following: If X is connected and Y is not, there exists no continuous surjection from X to Y.

Likewise, if X is path-connected but Y is not, there exists no continuous surjection from X to Y.

Exercise 22.2.6. Show that if X is connected, then for any equivalence relation $X/\sim$, the quotient space $X/\sim$ is connected.

Likewise, show that if X is path-connected, then for any equivalence relation $X/\sim$, the quotient $X/\sim$ is path-connected.

Exercise 22.2.7. Show that $\mathbb{R}P^2$ is path-connected and connected.

22.3 Connectedness is not path-connectedness

Last time we saw that if X is path-connected, then it is connected. We will see that the converse does not hold.

Definition 22.3.1 (Topologist's sine curve)**.** We let X be the following union:

$$\{(x_1, x_2) \mid x_1 = 0\} \bigcup \{(x_1, x_2) \mid x_1 > 0 \text{ and } x_2 = \sin(1/x_1)\ \} \subset \mathbb{R}^2.$$

We endow X with the subspace topology (inherited from $\mathbb{R}^2$). We call X the *topologist's sine curve.*

Exercise 22.3.2. Draw X (in $\mathbb{R}^2$).

Remark 22.3.3. The name "topologist's sine curve" is popular, but probably insinuates an immature separation of mathematical subjects. This space is no more a topologist's than anybody else's.

Theorem 22.3.4. *X is connected, but it is not path-connected.*

Before we prove the theorems, let's set some notation. We set

$$A = \{(x_1, x_2) \mid x_1 = 0\}$$

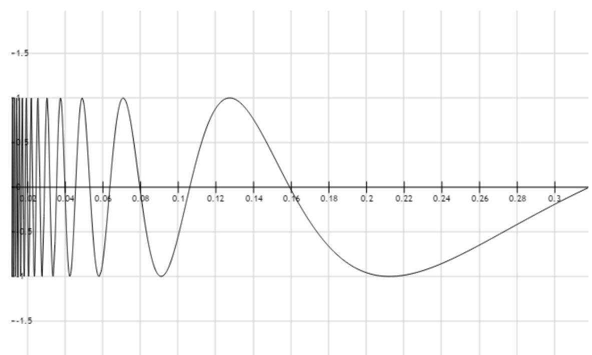
and

$$B = \{(x_1, x_2) \mid x_1 > 0 \text{ and } x_2 = \sin(1/x_1)\ \}$$

so we have that

$$X = A \cup B.$$

Remark 22.3.5. The following is what B looks like:



What happens as the x_1 coordinate approaches zero? One description might be that the x_2 coordinate oscillates—and it oscillates "faster and faster".

Lemma 22.3.6. *A and B are both homeomorphic to $\mathbb{R}$.*

Proof. I claim that the function

$$\mathbb{R} \to A, \qquad x \mapsto (0, x)$$

is a homeomorphism. I will leave the proof to you.

As for B, consider the maps

$$\mathbb{R}_{>0} \to B, \qquad t \mapsto (t, \sin(1/t))$$

and

$$B \to \mathbb{R}_{?0}, \qquad (x_1, x_2) \mapsto x_1.$$

These are both continuous and are inverses to each other. Moreover, $\mathbb{R}_{>0}$ is homeomorphic to $\mathbb{R}$ by taking, for example, the log and exp maps. This concludes the proof. □

Lemma 22.3.7. if U is an open subset of X and contains all of A, then it intersects B.

Proof. Let $U \subset X$ be open. If $A \subset U$, let $x = (0, 1) \in A$. Because U is an open subset of X, by definition of subspace topology, U is the intersection $W \cap X$ for some open subset $W \subset \mathbb{R}^2$. In particular, there must be some $\delta > 0$ so that the open ball of radius δ centered at x is contained in W. But given any δ¡ there exists some number positive $x_1' < \delta$ so that $1/x_1'$ is an integer multiple of $\pi/2$; in particular, there is some positive $x_1' < \delta$ so that

$$(x_1', \sin(1/x_1')) \in \mathrm{Ball}(x, \delta).$$

That is, $\mathrm{Ball}(x, \delta) \cap B$ is non-empty. This shows that for any $W \subset \mathbb{R}^2$ for which $W \cap X \supset A$, we have that $W \cap B \neq \emptyset$. That is, W intersects B, so U intersects B as well. □

Lemma 22.3.8. If K is a closed subset of X that contains all of B, then it must intersect A.

We will see a proof of this next time, when we discuss closures and closed subsetes of metric spaces.

Corollary 22.3.9. X is connected.

Proof. Let $Q \subset X$ be open and closed. Let us suppose Q is not empty. We are finished if we can show $Q = X$.

Because Q is non-empty, it contains some element x.

Let us suppose $x \in A$. Then $Q \cap A$ is non-empty; but because Q is both open and closed (in X), we conclude that $Q \cap A$ is both open in closed (in A). Because $A \cong \mathbb{R}$, A is connected; because $Q \cap A$ is non-empty, we conclude that $Q \cap A = A$. In other words, Q contains A. By Lemma 22.3.7, $Q \cap B$ is hence non-empty. But then $Q \cap B$ is a non-empty subset of B which is both open and closed; because B is connected (being homeomorphic to $\mathbb{R}$), we conclude that $Q \cap B = B$. But

$$X = A \cup B$$

so we conclude (using $Q \cap A = A$ and $Q \cap B = B$ with $Q \subset X$) that $X = Q$.

On the other hand, if $x \in B$, we again see that $Q \cap B = B$ by connectedness of B. By Lemma 22.3.8 we conclude $Q \cap A \neq \emptyset$, and thus $Q \cap A = A$ by connectedness of A. Hence $X = Q$. □

Now, to prove the theorem, it remains for us to prove that X is not path-connected. To that end, let us prove the following:

Lemma 22.3.10. Let $[t_0, t_1]$ be a closed interval. Then there does not exist a continuous function

$$f : [t_0, t_1] \to \mathbb{R}^2$$

such that $f(t_0) \in A$ and $f((t_0, t_1]) \subset B$.

Proof. We'll give a proof by contradiction, utilizing the "convergent sequence" criterion for continuity. (A continuous function preserves convergent sequences.)

Suppose a continuous f exists. Consider the composition

$$[t_0, t_1] \xrightarrow{f} \mathbb{R}^2 \xrightarrow{\pi_1 : (x_1, x_2) \mapsto x_1} \mathbb{R}$$

where the last arrow projects to the first coordinate. This composition is continuous (because both the above arrows are continuous).

Let us choose two sequences of real numbers. First, we choose a decreasing sequence

$$s_1, s_2, \ldots, \qquad s_i \in [t_0, t_1]$$

such that $\lim s_i \to t_0$, and so that $\sin(1/f(s_i))$ is constant.

(This can be constructed as follows: One first chooses an arbitrary $s_1 \in (t_0, t_1]$ (in particular, $s_1 \neq t_0$). By assumption, $\pi_1(f(s_1)) > 0$. If we have chosen s_i, by the continuity of f, we can find some

$$s_{i+1}$$

so that

1. $s_{i+1} \in (t_0, (s_i - t_0)/2]$, and

2. $\sin(1/f(s_{i+1})) = \sin(1/f(s_i))$.

By the first condition, the sequence s_i is decreasing and converges to t_0. The second condition ensures that the value $\sin(1/f(s_i))$ is constant with respect to i.)

We choose our second sequence

$$s'_1, s'_2, \ldots, \qquad s'_i \in [t_0, t_1]$$

again so the sequence is decreasing, so that $\lim s'_i \to t_0$, and so that $\sin(1/f(s'_i))$ is constant, but with the requirement that

$$\sin(1/f(s_i)) \neq \sin(1/f(s'_i)).$$

(This inequality can be achieved simply by a prudent choice of s'_1; we are using here that $\pi_2 \circ f$ is non-constant.)

But the composition

$$[t_0, t_1] \xrightarrow{f} \mathbb{R}^2 \xrightarrow{\pi_2 : (x_1, x_2) \mapsto x_2} \mathbb{R}$$

(where we now project to the second coordinate) is also continuous. Thus, we must have that

$$\sin(1/s_1) = \lim \pi_2 \circ f(s_i) = \pi_2 \circ f \lim s_i = \pi_2 \circ f(t_0)$$

(where the middle equality uses the continuity of $\pi_2 \circ f$) and, at the same time,

$$\sin(1/s'_1) = \lim \pi_2 \circ f(s'_i) = \pi_2 \circ f \lim s'_i = \pi_2 \circ f(t_0).$$

We arrive at a contradiction because $\sin(1/s_1) \neq \sin(1/s'_1)$. $\square$

Lemma 22.3.11. Let $a \in A$ and $b \in B$. There is no continuous path in X from a to b.

Proof. Let $\gamma : [0,1] \to X$ be continuous. Then $\gamma^{-1}(A) \subset [0,1]$ is a closed subset. (This is because $A \subset X$ is closed—to see this, note that $A \subset \mathbb{R}^2$ is closed.) On the other hand, $\gamma^{-1}(A) \neq [0,1]$ because $\gamma(1) = b \notin A$.

So let $t_0 = \max \gamma^{-1}(A)$ be the largest real number $t \in [0,1]$ for which $\gamma(t) \in A$. Then the composition

$$f : [t_0, 1] \to [0,1] \xrightarrow{\gamma}\to B \cup \{\gamma(t)\}$$

would be a continuous function contradicting the conclusion of Lemma 22.3.10. $\square$

Now we can finally prove the theorem:

Proof of Theorem 22.3.4. We know that X is not path-connected by Lemma 22.3.11. So it suffices to show that X is connected. This is the content of Corollary 22.3.9. □

22.4 Lessons learned

This lecture contained a lot of new mathematics. The reason we went in-depth was the following: I wanted to show you that a space can be connected, but not path-connected. The proofs above show that the "topologist's sine curve" is exactly such a space.

But there are other results we can glean from above.

Proposition 22.4.1. There exist topological spaces Y and continuous functions

$$f : (0, 1] \to Y$$

such that f does *not* extend to a continuous function on $[0, 1]$. that is, one can choose f and Y so that there does not exist a function

$$\gamma : [0, 1] \to Y$$

for which $\gamma(t) = f(t)$ for all $t \in (0, 1]$.

Indeed, even if we demand that $Y = \mathbb{R}^2$ and that f has bounded image, it is not always true that f extends to $[0, 1]$.

Proof. Let $Y = \mathbb{R}^2$. Take f to be the function

$$f(t) = (t, \sin(1/t)).$$

We saw that f does not extend continuously to a function $f : [0, 1] \to \mathbb{R}^2$.

As for the second part of the proposition, notice that the image of f is indeed bounded—for example, the image is contained in the rectangle $[0, 1] \times [-1, 1] \subset \mathbb{R}^2$, which is in turn contained in a ball of radius 3 centered at the origin. □

Remark 22.4.2. Another lesson learned is that the difference between "connected" and "path-connected" isn't too pathological.

("Pathological" is a term that mathematicians use to pass judgement on particular examples. A less judgmental, but equivalent, way to describe a "pathological example:" A pathological examples is one that betrays your early intuitions, and moreover, one having properties that we either rarely encounter, or want to avoid to make proving results easier.)

Note that the topologist's sine curve is Hausdroff—in fact, it's even a metric space (being a subspace of $\mathbb{R}^2$). These are the kinds of spaces that we thought we would feel somewhat comfortable with.

It depends on your tastes whether you want to interpret this example as saying ''Subspaces of $\mathbb{R}^2$ can be kind of crazy," or as saying "We should get used to certain phenomena because they will show up whether we expect them or not."

Lecture 23

Closures and interiors

Fix X a topological space. As you know, given a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of open subsets of X, the union

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$

is an open subset of X. Likewise, given a collection $\{K_\beta\}_{\beta \in \mathcal{B}}$ of closed subsets of X, the intersection

$$\bigcap_{\beta \in \mathcal{B}} K_\beta$$

is a closed subset of X.

Because of these properties, if we fix some subset $B \subset X$, it makes sense to speak of the "large open subset of X contained in B" and "the smallest closed subset of X containing B." These descriptions are informal; we'll make them precise shortly. They characterize the *interior* and *closure* of B, respectively.

These constructions (interiors and closures) are useful, and they're also fun and interesting. Fix some subset $B \subset \mathbb{R}^2$. It's not a bad use of one's day to figure out what the interior and closure of B are.

23.1 Closed subsets of metric spaces

Before we go on, let me prove the following:

Proposition 23.1.1. Let X be a metric space and fix a subset $A \subset X$. Then the following are equivalent

1. A is closed.

2. For every convergent sequence $x_1, \ldots$ such that $x_i \in A$ for every i, then the limit of the sequence is also in A.

Proof. You are proving (1) $\implies$ (2) in your homework. So here we'll prove the converse.

We'll prove (2) $\implies$ (1) by proving the contrapositive. So suppose that A is not closed. Then A^C is not open; so fix $y \notin A$ such that for every $r > 0$, $\text{Ball}(y, r) \not\subset A^C$. (At least one such y is guaranteed to exist if A^C is not open.)

Now fix a decreasing sequence of positive real numbers $r_1, r_2, \ldots$ converging to 0.[1] For every r_i, there exists some $x_i \in A \cap \text{Ball}(y, r_i)$. By construction, $x_1, x_2, \ldots$ is a sequence in A whose limit is y. This proves the contrapositive. □

23.2 Closure

Definition 23.2.1. Fix a topological space X and let $B \subset X$ be a subset.[2] Let

$$\mathcal{K}$$

be the collection of all closed subsets of X containing B.[3] Then the *closure* of B is defined to be

$$\overline{B} := \bigcap_{K \in \mathcal{K}} K.$$

In words, the closure of B is the set obtained by intersecting every closed subset containing B.

Remark 23.2.2. Note that B is always a subset of $\overline{B}$.

Remark 23.2.3. Note that $\overline{B}$ is a closed subset of X. This is because the intersection of closed subsets is always closed.

[1] For example, $r_i = 1/i$.

[2] It could be any kind of subset: open, closed, neither!

[3] Note that X is an element of $\mathcal{K}$.

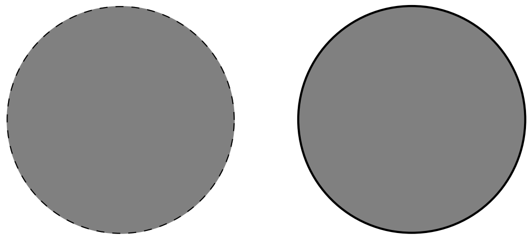


Figure 23.1: An open ball on the right; its closure (a closed ball) on the left.

Remark 23.2.4. If $B \subset X$ is closed, then $\overline{B} = B$. To see this, note that B is an element of $\mathcal{K}$ because B is closed. Hence

$$\bigcap_{K \in \mathcal{K}} K = B \cap \left(\bigcap_{K \in \mathcal{K}, K \neq B} K \right).$$

But this righthand side is a subset of B because it is obtained by intersecting B with some other set. In particular,

$$\overline{B} \subset B.$$

Because $B \subset \overline{B}$ (for any kind of B), we conclude that $B = \overline{B}$.

Example 23.2.5. If $B = \emptyset$, then $\overline{B} = \emptyset$. If $B = X$, then $\overline{B} = X$.

Exercise 23.2.6. Let $X = \mathbb{R}^n$ (with the standard topology). Let $B = \text{Ball}(0, r)$ be the open ball of radius r. Show that the closure of B is the closed ball of radius r; that is,

$$\overline{B} = \{x \in \mathbb{R}^n \text{ such that } d(x, 0) \leq r \ .\}$$

Proof. You are showing in your homework that if $K \subset X$ is closed and if $x_1, \ldots$ is a sequence in K converging to some $x \in X$, then x is in fact an element of K.

Choose a point x of distance r from the origin. And choose also an increasing sequence of positive real numbers $t_1, t_2, \ldots$ converging to 1.[4] Then the sequence

$$x_i = t_i x$$

[4] For example, you could take $t_i = i/(i + 1)$.

is a sequence in B converging to x. If $K \supset B$, then the x_i define a sequence in K; moreover, if K is closed, the limit x is in K. Thus $x \in K$ for any closed subset containing B. In particular, x is in the intersection of all such K. Thus $x \in \overline{B}$. This shows that the closed ball of radius r is contained in $\overline{B}$.

On the other hand, consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $d(0, -)$; that is, the "distance to the origin" function. We see that $f^{-1}([0, r])$ is equal to the closed ball of radius r—in particular, this closed ball is a closed subset of $\mathbb{R}^n$, and it obviously contains $\mathrm{Ball}(0, r)$. This shows that $\overline{B}$ is a subset of the closed ball of radius r (because $\overline{B}$ can be expressed as the intersection of this closed ball with other sets). We are finished. □

Exercise 23.2.7. Suppose $f : X \to Y$ is a continuous function, and let $B \subset X$ be a subset. Show that

$$f(\overline{B}) \subset \overline{f(B)}.$$

In English: The image of the closure of B is contained in the closure of the image of B.

Proof. Let $\mathcal{C}$ be the collection of closed subsets of Y containing $f(B)$. Then

$$f^{-1}(\overline{f(B)}) = f^{-1}\left(\bigcap_{C \in \mathcal{C}} C\right)$$

by definition of closure. We further have:

$$f^{-1}\left(\bigcap_{C \in \mathcal{C}} C\right) = \bigcap_{C \in \mathcal{C}} f^{-1}(C).$$

Now, because f is continuous, we know that $f^{-1}(C)$ is closed for every $C \in \mathcal{C}$. Moreover, because $f(B) \subset C$, we see that $B \subset f^{-1}(C)$. We conclude that for every $C \in \mathcal{C}$, $f^{-1}(C) \in \mathcal{K}$. Thus

$$\bigcap_{K \in \mathcal{K}} K \subset \bigcap_{C \in \mathcal{C}} f^{-1}(C).$$

The lefthand side is the definition of $\overline{B}$. The righthand side is $f^{-1}(\overline{f(B)})$. We are finished. □

Remark 23.2.8. It is not always true that $f(\overline{B})$ is equal to $\overline{f(B)}$. For example, let $B = X = \text{Ball}(0, r)$, and let $f : X \to \mathbb{R}^2$ be the inclusion. Then $f(\overline{B}) = X$, while $\overline{f(B)}$ is the closed ball of radius r.

Exercise 23.2.9. Find an example of a continuous function $p : \mathbb{R}^n \to \mathbb{R}$ such that
$$\overline{\{x \text{ such that } p(x) < t\}},$$
does *not* equal
$$\{x \text{ such that } p(x) \leq t\}.$$

Example 23.2.10. Let $B \subset \mathbb{R}^2$ be the following subset:
$$B = \{(x_1, x_2) \text{ such that } x_1 > 0 \text{ and } x_2 = \sin(1/x_1)\} \subset \mathbb{R}^2.$$
This is not a closed subset of $\mathbb{R}^2$. I claim
$$\overline{B} = B \bigcup \{(x_1, x_2) \text{ such that } x_1 = 0 \text{ and } x_2 \in [-1, 1]\}.$$
That is, $\overline{B}$ is equal to the topologist's sine curve from last class.

Let us call the righthand side S for the time being. First, I claim that $S \subset \overline{B}$. Indeed, fix some point $(0, T) \in S \setminus B$. Then there is an unbounded, increasing sequence of real numbers $t_1, t_2, \ldots$ for which $\sin(t_i) = T$; let $s_i = 1/t_i$. Then the sequence of points
$$x_i = (s_i, \sin(1/s_i)) = (s_i, T)$$
converges to $(0, T)$, while each x_i is an element of B. In particular, $(0, T)$ is contained in any closed subset containing B. This shows $S \subset \overline{B}$.

To complete the proof, it suffices to show that S is closed. For this, because $\mathbb{R}^2$ is a metric space, it suffices to show that any convergent sequence contained in S has a limit contained in S. So let $x_1, x_2, \ldots$ be a sequence in S.

Suppose that the limit $x \in \mathbb{R}^2$ has the property that the 1st coordinate is non-zero. There is a unique point in S with a given non-zero first coordinate t, namely $(t, \sin(1/t))$. Moreover, because the function $t \mapsto \sin(t/1)$ is continuous, if $t_i = \pi_1(x_i)$ converges to t, we know that $(t_i, \sin(1/t_i))$ converges to $(t, \sin(1/t))$. So the limit is in S.

If on the other hand the first coordinate of x is equal to zero, let us examine the second coordinates $\pi_2(x_1), \ldots$. By continuity of π_2, the sequence

$\pi_2(x_1), \pi_2(x_2), \ldots$ converges to some T; because each x_i has a second coordinate in $[-1,1]$, and because $[-1,1] \subset RR$ is closed, we conclude that the limit T is also contained in $[-1,1]$. Hence the limit of the sequence $x_1, \ldots,$ is the point $(0,T)$, and $(0,T) \in S$.

Because any sequence in S with a limit in $\mathbb{R}^2$ has limit in S, S is closed.

23.3 Interiors

Definition 23.3.1. Let X be a topological space and fix $B \subset X$. Let $\mathfrak{U}$ denote the collection of pen subsets of X that are contained in B. Then the *interior* of B is defined to be the union

$$int(B) = \bigcup_{U \in \mathfrak{U}} U.$$

Remark 23.3.2. For any B, we have that $int(B) \subset B$. Moreover, $int(B)$ is an open subset of both B and of X.

Remark 23.3.3. If B is open, then $int(B) = B$. This is because $B \in \mathfrak{U}$, so

$$int(B) = \bigcup_{U \in \mathfrak{U}} U = B \cup \left(\bigcup_{U \neq B, U \in \mathfrak{U}} U \right)$$

meaning $int(B)$ contains B (because $int(B)$ is a union of B with possibly other sets). Thus we have that $int(B) \subset B \subset int(B)$, meaning $int(B) = B$.

Example 23.3.4. We have that $int(\emptyset) = \emptyset$ and $int(X) = X$.

Example 23.3.5. Let $X = \mathbb{R}^n$ and let B be the closed ball of radius r. Then $int(B) = \text{Ball}(0,r)$ is the open ball of radius r.

To see this, we note that $\text{Ball}(0,r)$ is open and contained in B, so $\text{Ball}(0,r) \subset int(B)$ by definition of interior. Because $int(B) \subset B$, it suffices to show that no other point of B (i.e., no point in $B \setminus \text{Ball}(0,r)$) is contained in the interior of B.

So fix $y \in B \setminus \text{Ball}(0,r)$, meaning y is a point of exactly distance r away from the origin. It suffices to show that there is no open ball containing y and contained in B; for then there is no $U \in \mathfrak{U}$ for which $y \in U$.

Well, for any $\delta > 0$, $\text{Ball}(y,\delta) \subset \mathbb{R}^2$ contains some point of distance $> r$ from the origin. So $\text{Ball}(y,\delta)$ is never contained in B. This completes the proof.

Lecture 24

One-point compactification

Today you have a guest instructor. Get into your groups (you know the drill) and tackle the proofs of the propositions below.

Do not spent more than 15 minutes on proving a given proposition. Move on!

Definition 24.0.1. Let X be a topological space. We are now going to create a new topological space X^+.

As a set, $X^+ = X \coprod \{*\}$. In other words, X^+ is the set obtained by adjoining a single point called $*$ to X.

The topology $\mathcal{T}_{X^+}$ is defined as follows: $U \subset X^+$ is open if either

1. $* \notin U$ and U is open in X, or
2. $* \in U$ and $U \cap X$ is the complement of a closed, compact subspace of X.

We call X^+ the *one-point compactification* of X.

Remark 24.0.2. Note that if X is Hausdorff, we may remove the adjective "closed" from the second condition above.

24.1 Basic properties

Prove the following:

Proposition 24.1.1. $\mathcal{T}_{X^+}$ is a topology on the set X^+.

(Thus, you need to prove that the collection of sets U satisfying 1. or 2. satisfies all the properties of a topology. You will want to use at some point that the empty set is a compact space.)

Proposition 24.1.2. X^+ is compact.

(Thus, you need to prove that every open cover of X^+ admits a finite subcover.)

Remark 24.1.3. This justifies the word "compactification."

24.2 Examples

Prove the following:

Proposition 24.2.1. If X is compact, then X^+ is homeomorphic to the coproduct $X \coprod \{*\}$ with the coproduct topology.

Proposition 24.2.2. If $X = \mathbb{R}^n$, then X^+ is homeomorphic to S^n.

Remark 24.2.3. The proof of Proposition 24.2.2 is made easier if you know about *stereographic projection*. This is the function

$$p : S^n \setminus \{(0, \dots, 0, 1)\} \to \mathbb{R}^n, \qquad (x_1, \dots, x_{n+1}) \mapsto \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n).$$

Here is a description of p in words. For brevity, let us call the point $(0, \dots, 0, 1) \in S^n$ the *north pole* of S^n. Given a point $x \in S^n$ such that x is not the north pole, p sends x to the intersection of

- the line through x and the north pole, with
- the hyperplane $\{x_{n+1} = 0\}$, which one can identify with $\mathbb{R}^n$.

Proposition 24.2.4. If X and Y are homeomorphic, so are X^+ and Y^+.

Solutions to Lecture 24 Propositions

Proof of Proposition 24.1.1. (i) We first show $\emptyset, X^+$ is in this topology. So let $U = \emptyset$. Then $* \notin U$, so we must check whether $\emptyset$ is open in X (by condition 1 of the definition of $\mathcal{T}_{X^+}$). It is, by definition of topological space (i.e., because X itself is a topological space). Now let $U = X^+$. Since $* \in U$, we must check whether $U \cap X$ is the complement of a closed, compact subspace of X (by condition 2 of the definition of $\mathcal{T}_{X^+}$). It is, because $U \cap X = X$ and X is the complement of $\emptyset$. (Note that $\emptyset$ is both closed and compact.)

(ii) Now let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an arbitrary collection where $U_\alpha \in \mathcal{T}_{X^+}$ for any $\alpha \in \mathcal{A}$. We must show that the union

$$U := \bigcup_{\alpha \in \mathcal{A}} U_\alpha \subset X^+$$

is in $\mathcal{T}_{X^+}$.

Note that for any $\alpha \in \mathcal{A}$, we know that

$$U_\alpha \cap X$$

has a complement given by a closed subspace of X. (This is true regardless of whether U_α satisfies case 1. or in case 2. of the definition of $\mathcal{T}_{X^+}$.) Let us call this closed subspace K_α, and let us call the intersection

$$K := \bigcap_{\alpha \in \mathcal{A}} K_\alpha.$$

Note that the arbitrary intersection of closed subsets is closed, so $K \subset X$ is closed. Then by de Morgan's laws, we see that

$$X \cap U = X \cap \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha \right) = (\bigcap_{\alpha \in \mathcal{A}} K_\alpha)^C = K^C.$$

where the complement is taken inside X. Now, if $* \notin U$, then we have shown that U^C is closed, so by condition 1. of the definition of $\mathcal{T}^+$, we see that $U \subset X^+$ is indeed in $\mathcal{T}_{X^+}$.

On the other hand, if $* \in U$, then for some $\alpha \in \mathcal{A}$, we see that $* \in U_\alpha$. In particular, K_α is not only closed, but also compact. Thus $K \subset K_\alpha$ is a closed subspace of a compact K_α, meaning K itself is compact. This shows that $U \cap X = K^C$ is the complement of a compact, closed subspace of X, so U is open by condition 2. of the definition of $\mathcal{T}_{X^+}$.

(iii) Now we must show that a finite intersection of elements in $\mathcal{T}_{X^+}$ is in $\mathcal{T}_{X^+}$.

So fix $U_1, \ldots, U_n$, a finite collection of elements in $\mathcal{T}_{X^+}$. For each i, let $K_i = (U_i \cap X)^C$. Note that K_i is closed, and is compact if $* \in U_i$. We let

$$U = U_1 \cap \ldots \cap U_n \subset X^+$$

and

$$K = K_1 \cup \ldots \cup K_n \subset X.$$

Note that by de Morgan's laws, we again have

$$U \cap X = K^C \subset X$$

(where the complement is again taken inside X).

If $* \notin U$, then $U = K^C$. Being a complement of a closed subset in X, we see that $U \subset X$ is open in X, so $U \in \mathcal{T}_{X^+}$ by condition 1. of the definition.

If $* \in U$, then $* \in U_i$ for every i, so by condition 2, each K_i is not only closed but also compact. Lemma: The finite union of compact subspaces is compact. (Proof: Given an open cover of K, note that the open cover determines a finite subcover of each K_i. Taking the union of these finite subcovers, we have a finite union of finite collections; hence the resulting union is a finite open cover of K itself.) Thus K itself is compact. By condition 2, U is in $\mathcal{T}_{X^+}$. □

Proof of Proposition 24.1.2. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X^+. By definition of cover, there is some $\alpha_0 \in \mathcal{A}$ such that $* \in U_{\alpha_0}$. So by condition 2 of the definition of $\mathcal{T}_{X^+}$, we know

$$X^+ = U_{\alpha_0} \cup K$$

where K is a compact, closed subspace of X and $K \cap U_{\alpha_0} = \emptyset$.

Before we go any further, let us point out that $X \subset X^+$ is an open subset by condition 1 of the definition of $\mathcal{T}^+$. Thus the subspace topology of $K \subset X^+$ is equal to the subspace topology of $K \subset X$.

Invoking the definition of open cover, and by definition of subspace topology (for $K \subset X$), we know that the collection

$$\{U_\alpha \cap K\}_{\alpha in \mathcal{A}}$$

is an open cover of K. Since K is compact, we can choose some finite collection $\alpha_1, \ldots, \alpha_n$ so that $\{U_{\alpha_1} \cap K, \ldots, U_{\alpha_n} \cap K\}$ is an open cover of K. In particular,

$$U_{\alpha_0} \cup U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$$

is an open cover of X^+ itself. This exhibits a finite subcover of the original open cover, and we are finished. □

Proof of Proposition 24.2.1. We must show that $W \subset X^+$ is open if and only if $W \cap X$ and $W \cap \{*\}$ is open.

To see the latter claim, we must prove that the one-element set

$$U = \{*\} \subset X^+$$

is open. This is because $U \cap X = \emptyset = X^C$, where the complement is taken in X. But X is closed (as a subset of itself), and is compact by hypothesis, so by condition 2, U is open.

On the other hand, $W \cap X$ is always open for a one-point compactification—this is obvious if $* \notin W$ by condition 1, and if $* \in W$, then $W \cap X$ is a complement of a (compact and) closed subset of X by condition 2, hence by definition of closedness, $W \cap X$ is open in X.

This completes the proof. □

Proof of Proposition 24.2.2. Omitted, as it is entirely analogous to the solutions to homework. □

Proof of Proposition 24.2.4. Given a homeomorphism $f : X \to Y$, define a function

$$g : X^+ \to Y^+, \qquad x \mapsto \begin{cases} *_Y & x = *_X \\ f(x) & x \in X. \end{cases}$$

Here, $*_Y \in Y^+$ represents the "extra point" in the one-point-compactification of Y, and likewise for $*_X \in X^+$.

Clearly g is a bijection because f is. Let us show that $U \subset X^+$ is open if and only if $g(U) \subset Y^+$ is open.

1. If $*_X \notin U$, then $*_Y \notin g(U)$. But because f is a homeomorphism, $g(U) = f(U)$ is open if and only if $U \cap X = U$ is open.

2. If $*_X \in U$, then $*_Y \in g(U)$. This means that $U \cap X = K^C$ (where the complement is taken in X) for some compact, closed $K \subset X$. But because f is a homeomorphism, $K \subset X$ is compact and closed if and only if $f(K) \subset Y$

is also compact and closed. Thus $f(U) \cap Y$ is the complement of a closed, compact subspace of Y if and only if $U \cap X$ is the complement of a closed, compact subspace of X. This completes the proof. $\square$

Quiz: Nov 19, 2019

Write your name here:

Complete the following definitions:

1. Let X be a topological space. X is called *compact* if ....

2. Let X be a set. A collection $\mathcal{T}$ of subsets of X is called a *topology* if ...

Lecture 25

Density, Interiors

Today you have a guest instructor. Get into your groups (you know the drill) and tackle the proofs of the propositions below.

Do not spent more than 15 minutes on proving a given proposition. Move on!

25.1 Density

Definition 25.1.1. Let X be a topological space and fix a subset $B \subset X$. We say that B is *dense* in X if $\overline{B} = X$.

Prove the following:

Proposition 25.1.2. Fix $B \subset X$. The following are equivalent:

1. B is dense in X.
2. For every non-empty open $U \subset X$, $U \cap B \neq \emptyset$.
3. For every $x \in X$, and every neighborhood A of x in X, we have that $A \cap B \neq \emptyset$.
4. For every $x \in X$, and every open neighborhood A of x in X, we have that $A \cap B \neq \emptyset$.

Proposition 25.1.3. $\mathbb{Q} \subset \mathbb{R}$ is dense.

Proposition 25.1.4. $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 25.1.5. For each of the following examples of subsets of $\mathbb{R}^2$, identify the closure, the interior, and the boundary. Which of these is dense?

1. $B = \{(x_1, x_2)$ such that $x_1 \neq 0\ \}$.

2. $B = \bigcup_{(a,b)\in\mathbb{Z}\times\mathbb{Z}}(a-1, a+1) \times (b-1, b+1)$.

3. $B = \{(x_1, x_2)$ such that at least one of the coordinates is rational$\}$.

25.2 Interiors

Definition 25.2.1. Let X be a topological space and fix $B \subset X$. Let $\mathfrak{U}$ denote the collection of pen subsets of X that are contained in B. Then the *interior* of B is defined to be the union

$$int(B) = \bigcup_{U\in\mathfrak{U}} U.$$

Prove the following:

Proposition 25.2.2. For any B, we have that $int(B) \subset B$. Moreover, $int(B)$ is an open subset of both B and of X.

Proposition 25.2.3. $B \subset X$ is open if and only if $int(B) = B$.

Example 25.2.4. We have that $int(\emptyset) = \emptyset$ and $int(X) = X$.

Example 25.2.5. Let $X = \mathbb{R}^n$ and let B be the closed ball of radius r. Then $int(B)$ is the open ball of radius r.

To see this, we note that $\mathrm{Ball}(0, r)$ is open and contained in B, so $\mathrm{Ball}(0, r) \subset int(B)$ by definition of interior. Because $int(B) \subset B$, it suffices to show that no other point of B (i.e., no point in $B \setminus \mathrm{Ball}(0, r)$) is contained in the interior of B.

So fix $y \in B \setminus \mathrm{Ball}(0, r)$, meaning y is a point of exactly distance r away from the origin. It suffices to show that there is no open ball containing y and contained in B; for then there is no $U \in \mathfrak{U}$ for which $y \in U$.

Well, for any $\delta > 0$, $\mathrm{Ball}(y, \delta) \subset \mathbb{R}^2$ contains some point of distance $> r$ from the origin. So $\mathrm{Ball}(y, \delta)$ is never contained in B. This completes the proof.

Solutions to Lecture 25 Propositions

Proof of Proposition 25.1.2. There is a mistake in this problem: Condition 2 should say that for every *non-empty* $U \subset X$, we have $U \cap B \neq \emptyset$.

$1 \implies 2$. Proof by contrapositive. Suppose that there is some non-empty open $U \subset X$ such that $U \cap B = \emptyset$. Then U^C is closed while $U^C \supset B$, so the closure of B is contained in U^C by definition of closure. In particular, $\overline{B}$ does not contain U, so could not equal all of X.

$2 \implies 4$. This is obvious, as if A is an open neighborhood of x, then A is a non-empty open subset of X.

$4 \implies 3$. Given A a neighborhood of x, let $U \subset A$ be the open subset containing x (guaranteed by the definition of neighborhood). Then $U \cap B \neq \emptyset$ by 4, so $A \cap B \supset U \cap B \neq \emptyset$.

$3 \implies 1$. Clearly $\overline{B} \subset X$ always, so we must show that $X \subset \overline{B}$. Let $K \subset X$ be a closed subset containing B. Then K^C is open. If K^C is non-empty, choose $x \in K^C$, and note that K^C is a neighborhood of x. Thus by 3, $K^C \cap B \neq \emptyset$; this contradicts the fact that $B \subset K$. □

Proof of Proposition 25.1.3. Let $x \in \mathbb{R}$ be a real number, and for every integer $n \geq 1$, let x_n be any rational number in the interval $(x - 1/n, x + 1/n)$. Then the sequence x_n converges to x. By the sequence criterion for closure, we thus see that any real number is in the closure of $\mathbb{Q}$. □

Proof of Proposition 25.1.4. Same exact proof, except choose each x_n to be any *irrational* number in the interval $(x - 1/n, x + 1/n)$. □

Proof of Proposition 25.2.2. $int(B)$ is open in X because it is a union of open sets. (And unions of open sets are always open by definition of topology.) It is open in B because

$$int(B) \cap B = int(B),$$

and by definition of subspace topology, a subset of B is open if and only if it is an intersection of B with an open subset (like $int(B)$) of X.

Finally, $int(B) \subset B$ because $int(B)$ is a union of subsets of B. □

Proof of Proposition 25.2.3. If B is open, then obviously $B \in \mathfrak{U}$, while $U \in \mathfrak{U} \implies U \subset B$, so $\bigcup_{U \in \mathfrak{U}} U \subset B$ while $B \subset \bigcup_{U \in \mathfrak{U}} U$. Hence $B = int(B)$.

On the other hand, if $B = int(B)$, then B is a union $\bigcup_{U \in \mathfrak{U}} U$ of open subsets of X; hence B is an open subset of X. □

Quiz: Nov 21, 2019

Write your name here:

Complete the following definitions:

1. Let X and Y be topological spaces. A function $f : X \to Y$ is called a *homeomorphism* if ...

2. Let X be a topological space. X is called *path-connected* if ...

Lecture 26

Some fun examples

Do not spent more than 10 or 15 minutes on proving a given proposition. Move on and have fun!

26.1 Euclidean space is an open ball

Here is a basic one:

Proposition 26.1.1. Let $r > 0$ and fix $x \in \mathbb{R}^n$. Then $\mathrm{Ball}(x, r)$ is homeomorphic to $\mathbb{R}^n$.

26.2 Tori

Here are four spaces:

1. A is the product space $S^1 \times S^1$.

2. B is the quotient space $[0,1] \times [0,1]/\sim$, where $\sim$ is the following equivalence relation:

$$x \sim x' \iff \begin{cases} x_1, x_1' \in \{0,1\} \text{ and } x_2 = x_2' & \text{or} \\ x_1 = x_1' \text{ and } x_2, x_2' \in \{0,1\} \end{cases}.$$

   Here, $x = (x_1, x_2) \in [0,1] \times [0,1]$. (It may help to draw a picture on a square showing what kind of "gluing" is happening.)

3. C is the surface in $\mathbb{R}^3$ parametrized by the equation

$$\vec{r}(\theta, \phi) = ((a\cos\theta + b)\cos\phi, (a\cos\theta + b)\sin\phi, a\sin\theta)$$

for $0 < a < b$ and $\theta, \phi \in [0, 2\pi]$.

4. D is the subset in $\mathbb{C} \times \mathbb{C}$ given by points $(x, y) \in \mathbb{C} \times \mathbb{C}$ such that $|x| = |y| = 1$ (given the subspace topology).

Proposition 26.2.1. A and D are homeomorphic.

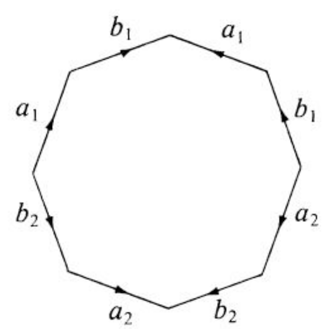
Proposition 26.2.2. B and C are homeomorphic.

Proposition 26.2.3. Any two of the spaces in $\{A, B, C, D\}$ are homeomorphic.

Proposition 26.2.4. If X is any of the above spaces, then for every $x \in X$, there is some open subset $U \subset X$ with $x \in U$ such that U is homeomorphic to $\mathbb{R}^2$.

26.3 What the...?

Can you draw what happens when you glue together the edges of an octahedron as below? (You glue a_1 to the other edge labeled a_1, in the way respecting directions as indicated. Likewise glue edge a_2 to a_2, and b_1 to b_1, and b_2 to b_2.)



Proposition 26.3.1. Let X be the space obtained by gluing as above. For every $x \in X$, there is some open subset $U \subset X$ with $x \in U$ such that U is homeomorphic to $\mathbb{R}^2$.

Solutions to Lecture 26 Propositions

Proof of Proposition 26.1.1. Step 0. Note that $\mathrm{Ball}(x, r)$ is homeomorphic to $\mathrm{Ball}(0, r)$. This is by translation:

$$\mathrm{Ball}(x, r) \to \mathrm{Ball}(0, r), \qquad y \mapsto y - x.$$

The inverse map is given by $y \mapsto y + x$. These are both continuous functions because addition of a constant (called x) is a continuous function.

Step 1. Note that $\mathrm{Ball}(0, r)$ is homeomoprhic to $\mathrm{Ball}(0, 1)$. This is because the function

$$\mathrm{Ball}(0, r) \to \mathrm{Ball}(0, 1), \qquad y \mapsto y/r$$

is clearly continuous (it is polynomial, we're scaling y by a constant). The inverse function is given by $y' \mapsto ry'$, and is again continuous.

Step 2. We must show that $\mathrm{Ball}(0, 1)$ is homeomorphic to $\mathbb{R}^n$. Then the composition

$$\mathrm{Ball}(x, r) \xrightarrow{step0} \mathrm{Ball}(0, r) \xrightarrow{step1} \mathrm{Ball}(0, 1) \to \mathbb{R}^n$$

is a homeomorphism we seek.

First note that the "distance to the origin" function

$$\mathbb{R}^n \to \mathbb{R}_{\geq 0} \qquad y \mapsto d(y, 0)$$

is continuous from a previous in-class exercise. In particular, the composition

$$\mathrm{Ball}(0, 1) \to \mathbb{R}^n \to [0, \pi/2] \qquad y \mapsto \pi/2 \cdot d(y, 0)$$

(called "include the ball into $\mathbb{R}^n$," then "measure distance to origin," then "scale") is continuous. Let us now post-compose with the arctan function, which is also continuous:

$$\mathrm{Ball}(0, 1) \to \mathbb{R}^n \to [0, \pi/2] \to [0, \infty) y \mapsto \arctan(\pi/2 \cdot d(y, 0)).$$

We will call this composition f, so $f(y) = \arctan(\pi/2 \cdot d(y, 0))$. Then consider the function

$$\phi : \mathrm{Ball}(0, 1) \to \mathbb{R}^n, \qquad y \mapsto f(y) \cdot y.$$

Geometrically, this keeps the vector y in "the same direction," but scales the length of Y.

ϕ is continuous because it is the product of two continuous functions called f and $\mathrm{id}_{\mathbb{R}^n}$. It is straightforward to check that ϕ is a bijection by checking that the function $[0,1] \to [0,\infty), \qquad t \mapsto \arctan(\pi/2t)$ is a bijection. An inverse is given by sending

$$y \mapsto \tan(d(y,0))y \cdot 2/\pi.$$

□

Proof of Proposition 26.2.1. By definition, $S^1 \subset \mathbb{R}^2$, so we can see that $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{C} \times \mathbb{C}$. It remains to see that S^1 is equal exactly to those $x \in \mathbb{R} \times \mathbb{R} \cong \mathbb{C}$ having norm 1.

To be more rigorous, I needed to tell you that I identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, and endow $\mathbb{C}$ with the exact same topology as I endowed $\mathbb{R}^2$. I also need to argue that the product topology $S^1 \times S^1$ coincides with the subspace topology of seeing $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$, but I will omit this. □

Proof of Proposition 26.2.2. Note the function

$$l : [0,1] \times [0,1] \to [0, 2\pi \times [0, 2\pi], \qquad (s,t) \mapsto (2\pi s, 2\pi t)$$

given by scaling. This is obviously a homeomorphism with inverse given by $(\theta, \phi) \mapsto 1/2\pi(\theta, \phi)$. Then the composition

$$[0,1] \times [0,1] \xrightarrow{l} [0, 2\pi \times [0, 2\pi] \xrightarrow{\vec{r}} \mathbb{R}^3$$

where $\vec{r}$ is the function defining C, is a surjection onto C. Moreover, this composition is continuous because l and $\vec{r}$ are. Now let $\sim$ be the equivalence relation on $[0,1] \times [0,1]$ given by $x \sim x' \iff \vec{r} \circ l(x) = \vec{r} \circ l(x')$. By analyzing $\vec{r}$, you can check this relation is precisely the relation defining B.

Since the composition is continuous, by a result from homework, the map out of the quotient

$$g : B \to C$$

is continuous. Because $[0,1] \times [0,1]$ is compact, so is B (being a quotient of a compact space). Moreover, C is a subspace of $\mathbb{R}^3$, so is Hausdorff. But g is continuous by our previous steps, and is precisely a bijection because of the equivalence relation defining B. Since B is compact and C is Hausdorff, we conclude that g is a homeomorphism. □

Proof of Proposition 26.2.3. It suffices to exhibit a homeomorphism from B to D. To do this, consider the function

$$j : [0, 1] \times [0, 1] \to \mathbb{C} \times \mathbb{C}, \qquad (s, t) \mapsto (e^{2\pi is}, e^{2\pi it}).$$

(In case you haven't seen this before, $e^{is} = \cos(2\pi s) + i\sin(2\pi s)$.) This is clearly continuous and a surjection onto D. You can check that the equivalence relation $(s, t) \sim (s', t') \iff j(s, t) = j(s', t')$ is precisely the same as the relation defining B. By the same reasoning as in the previous proof, we see that j is a continuous bijection, that B is compact, and that D is Hausdorff; hence j is a homeomorphism. □

Proof of Proposition 26.2.4. Omitted. □

Proof of Proposition 26.3.1. Omitted. □

Lecture 27

More fun examples

Today's goal is to see examples of different topologies on the same set (just as a single set may admit different metrics). In fact, you've already seen instances of this phenomenon in the trivial and discrete topologies.

Do not spend more than 10 or 15 minutes on proving a given proposition. Move on and have fun!

27.1 Two points!

Let $A = \{a, b\}$ be a set with exactly two elements. (As indicated in the previous sentence's notation, we call the two elements a and b.)

Prove the following propositions:

Proposition 27.1.1. *A admits exactly four topologies.*

(That is, there are exactly four choices of $\mathcal{T}$. To get started, it may help to begin with the *trivial* topology and the *discrete* topology. To find the other two, of course, you should think hard about the definition of what a topology is.)

Proposition 27.1.2. *Out of these four topologies on A, exactly two of them are homeomorphic.*

So out of the four topologies, two are equivalent, meaning there are roughly three "kinds" of different topologies on A. We say that, "up to homeomorphism," there are exactly three topologies on a two-element set.

27.2 Arr?

You are about to explore *upper semicontinuous* functions and *lower semicontinuous* functions. As it turns out, a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is both lower and upper semicontinuous. (See Proposition 27.2.9 below.) This is the historical reason for which these "semicontinuity" notions were studied—it's sometimes easier to prove upper and lower semicontinuity than to prove continuity.

Definition 27.2.1. We define a topology $\mathcal{T}_{\text{upper}}$ on $\mathbb{R}$ as follows: A subset $U \subset \mathbb{R}$ is in $\mathcal{T}_{\text{upper}}$ if and only if

1. U is empty,
2. $U = \mathbb{R}$, or
3. $U = (-\infty, a)$ for some $a \in \mathbb{R}$.

$\mathcal{T}_{\text{upper}}$ is called the *upper semicontinuous topology* on $\mathbb{R}$.

Prove:

Proposition 27.2.2. $\mathcal{T}_{\text{upper}}$ is indeed a topology on X.

You may skip the following if you like:

Exercise 27.2.3. For every real number $r \in \mathbb{R}$, let $U_r = (-\infty, r)$. Then the intersection

$$\bigcap_{r>0} U_r$$

is equal to $(-\infty, 0]$. In particular, the intersection is not in $\mathcal{T}_{\text{upper}}$ (even though each U_r is).

This shows that $\mathcal{T}_{\text{upper}}$ is not closed under arbitrary intersections.

Remark 27.2.4. Note that $\mathcal{T}_{\text{upper}}$ is a subcollection of, and not equal to, the standard topology $\mathcal{T}_{std}$ on $\mathbb{R}$. Note also that $(\mathbb{R}, \mathcal{T}_{\text{upper}})$ is not a Hausdorff topological space, so the topology $\mathcal{T}_{\text{upper}}$ does not arise from any metric.

Definition 27.2.5. Consider a function $f : \mathbb{R} \to \mathbb{R}$. f is called *upper semicontinuous* if f is continuous as a function from $(\mathbb{R}, \mathcal{T}_{std})$ to $(\mathbb{R}, \mathcal{T}_{\text{upper}})$.

Prove:

Proposition 27.2.6. Define functions g and $h : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases}, \qquad h(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0. \end{cases}.$$

Then

1. g and h are *not* continuous as a function from $(\mathbb{R}, \mathcal{T}_{std})$ to $(\mathbb{R}, \mathcal{T}_{std})$. (Do not spent much time on this; you already know this!)

2. g is upper semicontinuous.

3. h is not upper semicontinuous.

Definition 27.2.7. We define a topology $\mathcal{T}_{\text{lower}}$ on $\mathbb{R}$ as follows: A subset $U \subset \mathbb{R}$ is in $\mathcal{T}_{\text{lower}}$ if and only if

1. U is empty,

2. $U = \mathbb{R}$, or

3. $U = (a, \infty)$ for some $a \in \mathbb{R}$.

$\mathcal{T}_{\text{lower}}$ is called the *lower semicontinuous topology* on $\mathbb{R}$.

A function $f : \mathbb{R} \to \mathbb{R}$ is called *lower semicontinuous* if f is continuous as a function from $(\mathbb{R}, \mathcal{T}_{std})$ to $(\mathbb{R}, \mathcal{T}_{\text{lower}})$.

You may skip the following exercise if you wish:

Exercise 27.2.8. Let g and h be the functions above.

1. g is not lower semicontinuous.

2. h is lower semicontinuous.

Prove:

Proposition 27.2.9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the following are equivalent:

1. f is continuous as a function from $(\mathbb{R}, \mathcal{T}_{std})$ to $(\mathbb{R}, \mathcal{T}_{std})$.

2. f is both upper and lower semicontinuous.

You may skip the following if you wish:

Exercise 27.2.10. Define $\mathcal{S}$ as follows: A subset $B \subset \mathbb{R}$ is in $\mathcal{S}$ if and only if: B is empty, $B = \mathbb{R}$, or $B = [a, \infty)$ for some $a \in \mathbb{R}$.

Then $\mathcal{S}$ is *not* a topology on $\mathbb{R}$.