1 Solutions: Metric spaces

1.1 Definitions

You should know how to define:

1. What a metric (on a set X) is.
2. What a metric space is.
3. What a continuous function between metric spaces is (using ϵ and δ).
4. What a convergent sequence in a metric space is.
5. What an open ball (centered at x of radius r) is.
6. What an open set of a metric space is.
7. The topology associated to a metric.
8. The discrete metric on a set X.
9. The taxicab, standard, and l^∞ metrics on $\mathbb{R}^n$.

1. A metric on X is a function $d : X \times X \to \mathbb{R}$ satisfying the following for any $x, x', x'' \in X$: (0) $d(x, x') = 0$ if and only if $x = x'$. (1) $d(x, x') = d(x', x)$, and (2) $d(x, x') + d(x', x'') \geq d(x, x'')$.

2. A metric space is a pair (X, d) where X is a set and d is a metric on X.

3. A function $f : X \to Y$ is continuous if and only if the following holds: For every $\epsilon > 0$ and for every $x \in X$, there exists $\delta > 0$ such that whenever $d_X(x, x') < \delta$, we have that $d_Y(f(x), f(x')) < \epsilon$.

4. A sequence in X is a choice of element $x_i \in X$ for every positive integer i. For a point $x \in X$, say that the sequence $x_1, x_2, \ldots$ *converges to* x if for all $\epsilon > 0$, there exists an integer N so that $i \geq N \implies d_X(x_i, x) < \epsilon$. We say that a sequence converges if it converges to some $x \in X$.

5. An open ball of radius r and centered at x is the set of all points $x' \in X$ such that $d(x, x') < r$.

6. An open subset of a metric space (X, d) is any subset $U \subset X$ that be written as a union of open balls.

7. Given a metric d on X, we declare $\mathcal{T}_X$ to consist of open subsets (in the above sense). We showed in homework and in class that $\mathcal{T}_X$ is indeed a topology on X—it contains $\emptyset$ and X, it is closed under finite intersection, and it is closed under arbitrary unions.

8. Given a set X, the discrete metric is the metric defined by

$$d(x, x') = \begin{cases} 0 & x = x' \\ 1 & x \neq x' \end{cases}$$

9. These are defined by

$$d_{taxi}(x, x') = |x'_1 - x_1| + \ldots + |x'_n - x_n| = \sum_{i=1}^{n} |x'_i - x_i|$$

$$d_{std}(x, x') = \sqrt{(x'_1 - x_1)^2 + \ldots + (x'_n - x_n)^2} = \sqrt{\sum_{i=1}^{n} (x'_i - x_i)^2}$$

$$d_{l^\infty}(x, x') = \max\{|x'_1 - x_1|, \ldots, |x'_n - x_n|\}.$$

1.2

Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \to Y$ is an isometry. Prove that the inverse function $g : Y \to X$ is also an isometry.

First let us note that if f is an isometry, then its inverse function g is also an isometry. To see this, for any $y, y' \in Y$, let $x = g(y)$ and $x' = g(y')$. Then

$$d_X(g(y), g(y')) = d_X(gf(x), gf(x')) = d_X(x, x') = d_Y(y, y').$$

This proves g is an isometry.

1.3

Give an example of two metric spaces (X, d_X) and (Y, d_Y), along with a continuous bijection $f : X \to Y$, such that the inverse function $g : Y \to X$ is not continuous.

Let $X = Y = \mathbb{R}^n$ and $d_X = d_{discrete}, d_Y = d_{std}$. Let f be the identity function. See now solutions to 1.8 and 1.7.

1.4

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be an isometric embedding. Show that f is continuous.

Given $\epsilon > 0$ and $x \in X$, let $\delta = \epsilon$. Because f is an isometric embedding, we know that for all $x, x' \in X$,

$$d_Y(f(x), f(x')) = d_X(x, x').$$

So if $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \delta = \epsilon$.

1.5 (This one is a little involved)

Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous function. Let

$$G := \{(x, y) \text{ such that } f(x) = y\ \} \subset X \times Y$$

and let

$$U := X \times Y \setminus G$$

denote the complement of G. Show that U is an open subset of $X \times Y$. (Here, we are giving $X \times Y$ the product metric.)

We must prove that U is a union of open balls in $X \times Y$. As we saw in class, it suffices to show that for any element $(x, y) \in U$, we must prove the existence of some $r > 0$ for which

$$\mathrm{Ball}_{X\times Y}((x,y);r) \subset U. \tag{1.1}$$

So given (x, y), let $\epsilon = d_Y(y, f(x))/2$. Note that $\epsilon > 0$ because $y \neq f(x)$ (by definition of U).
On the other hand, by continuity of f, there exists some $\delta > 0$ so that if $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \epsilon$.
So let $r = \min\{\epsilon, \delta\}$. We aim to show (1.1). So let $(x', y') \in \mathrm{Ball}((x, y); r)$. Indeed, applying the triangle inequality repeatedly, we find

$$\begin{aligned} d_Y(f(x), f(x')) + d_Y(f(x'), y') + d_Y(y', y) &\geq d_Y(f(x), y') + d_Y(y', y) \\ &\geq d_Y(f(x), y) \\ & 2\epsilon. \end{aligned}$$

Thus we find

$$d_Y(f(x'), y') \geq 2\epsilon - d_Y(y', y) - d_Y(f(x), f(x')). \tag{1.2}$$

But we know that $d_Y(y', y) < \epsilon$ because of the definition of the product metric: We know that

$$d_{X\times Y}((x,y),(x',y')) = d_X(x,x') + d_Y(y,y') < r = \min\{\epsilon,\delta\} \leq \epsilon$$

so $d_Y(y, y') < \epsilon$ (because we know $d_X(x, x') \geq 0$). Likewise, we know that $d_X(x, x') < \delta$, so we must have that $d_Y(f(x), f(x')) < \epsilon$. In other words, the two negative terms on the righthand side of (1.2) both have absolute values strictly less than ϵ. This means that the righthand side of (1.2) is positive. This shows that $d_Y(f(x'), y') \neq 0$; in particular, $f(x') \neq y'$ because d_Y is a metric. This shows that (x', y') is in U, which was to be shown.

1.6 Open balls for various metrics

Given $x \in \mathbb{R}^2$ and $r > 0$, you should be able to draw

1. The open ball centered at x, with radius r, with respect to the standard metric d_{std}.

2. The open ball centered at x, with radius r, with respect to the taxicab metric d_{taxi}.

3. The open ball centered at x, with radius r, with respect to the l^∞ metric d_{l^∞}.

4. The open ball centered at x, with radius r, with respect to the discrete metric d_{disc}.

1. By the definition of standard metric, $\text{Ball}(x, r)$ consists of those points x' that are distance less than r from x (under the usual notion of distance). Thus $\text{Ball}(x, r)$ consists of those points that are on the interior of a circle of radius r (the circle is not included in its interior).

2. By the definition of taxi cab metric, $\text{Ball}(x, r)$ consists of those x' such that the sum $|x'_1 - x_1| + |x'_2 - x_2|$ is strictly less than r. Let us first assume that $x'_1 - x_1$ and $x'_2 - x_2$ are both non-negative, so that we are studying the set of x'_1, x'_2 satisfying

$$(x'_1 - x_1) + (x'_2 - x_2) < r \qquad \text{and} x'_1 \geq x_1 \qquad \text{and} \qquad x'_2 \geq x_2.$$

You learned how to draw the region of points (x'_1, x'_2) satisfying the above inequalities in a previous class: To address the first inequality, one first draws the line $x'_1 + x'_2 = (r + x_1 + x_2)$—here x'_1, x'_2 are variables while x_1, x_2 are fixed—and considers all points under this line. The set of points satisfying the second inequality is the half-plane lying to the right of the vertical line with constant horizontal coordinate x_1. The set of points satisfying the third inequality is given by the half-plane lying above the horizontal line with constant vertical coordinate x_2. The intersection of these three regions gives the a right triangle T and its interior, excluding its hypotenuse.

One can likewise study the regions given by

$$(x'_1 - x_1) - (x'_2 - x_2) < r \qquad \text{and} x'_1 \geq x_1 \qquad \text{and} \qquad x'_2 \leq x_2.$$

$$-(x'_1 - x_1) + (x'_2 - x_2) < r \qquad \text{and} x'_1 \leq x_1 \qquad \text{and} \qquad x'_2 \geq x_2.$$

$$-(x'_1 - x_1) - (x'_2 - x_2) < r \qquad \text{and} x'_1 \leq x_1 \qquad \text{and} \qquad x'_2 \leq x_2.$$

The union of the four regions studied gives the open ball of radius r about (x_1, x_2). The same way that we obtained a right triangle (with hypotenuse removed) for the first region we studied, you ca check that each of the three remaining are right triangles (and their interiors, though with hypoteneuse removed), and these right triangles are obtained by rotating or reflecting the first triangle T we studied. You can check that the union of these four regions indeed gives a diamond shape, centered at (x_1, x_2), with vertical and horizontal axes of length $2r$.

3. For the l^∞ metric: Fix $x \in \mathbb{R}^2$ and $r > 0$. Then a point x' is contained in $\text{Ball}_{taxi}(x, r)$ if and only if

$$\max\{|x'_1 - x_1|, |x'_2 - x_2|\} < r.$$

Put another way, we must have that

$$|x'_1 - x_1| < r \qquad \text{and} \qquad |x'_2 - x_2| < r.$$

So let

$$A = \{(x'_1, x'_2) \text{such that } |x'_1 - x_1| < r\} \qquad \text{and} \qquad B = \{(x'_1, x'_2) \text{such that } |x'_2 - x_2| < r\}.$$

So that the open ball we seek is the intersection of A and B.

A can be drawn as an infinitely long vertical strip: This is because A poses no restriction on the vertical coordinate x'_2 of x', but the horizontal coordinate x'_1 must be contained in the open interval $(x_1 - r, x_1 + r)$. Likewise, B can be drawn as an infinitely long horizontal strip.

The intersection of A and B is now easily checked to be a square (without its boundary), centered at x, and of width $2r$.

4. For the discrete metric, we get very different answers depending on r. If $r \leq 1$, then the only point x' such that $d_{discrete}(x, x') < r < 1$ is given by $x' = x$. Hence

$$\text{Ball}(x, r) = \{x\}.$$

For example, the open ball of radius 0.5 centered at x is a set with only one element: x itself.

On the other hand, if $r > 1$, then any point x satisfies $d_{discrete}(x, x') < r$, because $d_{discrete}$ only takes on the values 0 and 1. In other words,

$$\text{Ball}(x, r) = \mathbb{R}^2.$$

So for example, the open ball of radius 1.5 (centered at any x) is all of $\mathbb{R}^2$.

1.7

Prove that the identity function

$$id : (\mathbb{R}^2, d_{disc}) \to (\mathbb{R}^2, d_{std})$$

is continuous.

Regardless of x and ϵ, choose δ to equal 0.5 (or any number less than 1). Then if $d_{discrete}(x, x') < \delta$, we know that $x = x'$. In particular, $d_{std}(id(x), id(x')) = d_{std}(x, x') = d_{std}(x, x) = 0 < \epsilon$.

1.8

Prove that the identity function

$$id : (\mathbb{R}^2, d_{std}) \to (\mathbb{R}^2, d_{disc})$$

is not continuous.

Recall that to show a function $f : (X, d_X) \to (Y, d_Y)$ is *not* continuous, we have to show the following: There exists an $\epsilon > 0$ and an $x \in X$ so that, for any $\delta > 0$, there exists x' with $d_X(x, x') < \delta$ for which $d_Y(f(x), f(x')) \geq \epsilon$. We let $\epsilon = 0.5$ (or any number less than 1); it turns out our choice of x does not matter. Then note that $d_{disc}(y, y') < \epsilon$ if and only if $y = y'$. However, for any $\delta > 0$, there exist $x' \neq x$ for which $d_{std}(x, x') < \delta$; and for such x' we have that $d_{disc}(id(x), id(x')) = d_{disc}(x, x') = 1 \not< \epsilon$.

1.9 Open subsets, I

Verify the following:

1. The open interval $(-5, 5)$ is open in $(\mathbb{R}, d_{std})$.

2. The set $\mathbb{R}$ is open in $(\mathbb{R}, d_{std})$.

3. The set $U = \mathbb{R} \setminus \mathbb{Z}$ (of all numbers that are not integers) is open in $(\mathbb{R}, d_{std})$.

4. The set $U = \mathbb{R}^2 \setminus \mathbb{Z}^2$ (of all points in the plane whose coordinates are not both integers) is open in $(\mathbb{R}^2, d_{std})$.

(1) This is the open ball of radius 5 centered at the origin. Any open ball B is a union of open balls (in fact, B is a union of a single open ball: B itself), so we are finished.
(2) In any metric space (X, d_X), X is open. This is because, for example $X = \bigcup_{x \in X, r>0} \mathrm{Ball}(x, r)$.
(3) Let $x \in U$. (So x is any non-integer real number.) Then there is a closest integer N to x. Let $r = |x - N|$. Then $B(x, r) \subset U$ and $x \in B(x, r)$. From class, we saw that a subset $U \subset X$ is open if and only if for every $x \in U$, there exists $r > 0$ such that $B(x, r) \subset U$. That's what we've shown. So U is open.
(4) Let $x = (x_1, x_2) \in U$. then there is a closest integer N_1 to x_1, and a closest integer N_2 to x_2. Let $r = \min\{|x_1 - N_1|, |x_2 - N_2|\}$. Then $B(x, r) \subset U$. So U is open.

1.10 Open subsets, II

Verify the following:

1. The set $\mathbb{Q}$ (of rational numbers) is not open in $(\mathbb{R}, d_{std})$.
2. The set $\mathbb{R} \setminus \mathbb{Q}$ (of irrational numbers) is not open in $(\mathbb{R}, d_{std})$.
3. The set $\mathbb{R}^2 \setminus S^1$ (of all points not on the unit circle) is open in $(\mathbb{R}^2, d_{std})$.

(1) Let $x \in \mathbb{Q}$ be a rational number. Then for any $r > 0$, there is an irrational number within distance r of x. Hence for any $r > 0$, $B(x, r)$ is *not* contained in $\mathbb{Q}$. Hence $\mathbb{Q}$ is not open.
(2) Same proof as above—for any irrational number x, and any $r > 0$, there exists a rational number within distance x of r.
(3) Let $U = \mathbb{R}^2 \setminus S^1$ and $x \in U$. Also let $r = d(x, 0)$ where 0 is the origin in $\mathbb{R}^2$. If $r < 1$, then $B(x, 1-r) \subset U$. This is because for any $x' \in S^1$, we have that

$$1 = d(0, x') \leq d(0, x) + d(x, x') = r + d(x, x').$$

Thus $1-r \leq d(x, x')$. In particular, $x \notin B(x, 1-r)$; this means $B(x, 1-r) \subset U$.
If $r > 1$, then for any $x' \in S^1$, we have that

$$r = d(0, x) \leq d(0, x') + d(x, x') = 1 + d(x, x')$$

so $d(x, x') \geq 1 - r$, and in particular, $x' \notin B(x, 1-r)$. This means $B(x, 1-r) \subset U$, so U is open.