Practice problems

1 Metric spaces

1.1 Definitions

You should know how to define:

1. What a metric (on a set X) is.
2. What a metric space is.
3. What a continuous function between metric spaces is (using ϵ and δ).
4. What a convergent sequence in a metric space is.
5. What an open ball (centered at x of radius r) is.
6. What an open set of a metric space is.
7. The topology associated to a metric.
8. The discrete metric on a set X.
9. The taxicab, standard, and l^∞ metrics on $\mathbb{R}^n$.

1.2

Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \to Y$ is an isometry. Prove that the inverse function $g : Y \to X$ is also an isometry.

1.3

Give an example of two metric spaces (X, d_X) and (Y, d_Y), along with a continuous bijection $f : X \to Y$, such that the inverse function $g : Y \to X$ is not continuous.

1.4

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be an isometric embedding. Show that f is continuous.

1.5 (This one is a little involved)

Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous function. Let

$$G := \{(x, y) \text{ such that } f(x) = y \} \subset X \times Y$$

and let

$$U := X \times Y \setminus G$$

denote the complement of G. Show that U is an open subset of $X \times Y$. (Here, we are giving $X \times Y$ the product metric.)

1.6 Open balls for various metrics

Given $x \in \mathbb{R}^2$ and $r > 0$, you should be able to draw

1. The open ball centered at x, with radius r, with respect to the standard metric d_{std}.
2. The open ball centered at x, with radius r, with respect to the taxicab metric d_{taxi}.
3. The open ball centered at x, with radius r, with respect to the l^∞ metric d_{l^∞}.
4. The open ball centered at x, with radius r, with respect to the discrete metric d_{disc}.

1.7

Prove that the identity function

$$id : (\mathbb{R}^2, d_{disc}) \to (\mathbb{R}^2, d_{std})$$

is continuous.

1.8

Prove that the identity function

$$id : (\mathbb{R}^2, d_{std}) \to (\mathbb{R}^2, d_{disc})$$

is not continuous.

1.9 Open subsets, I

Verify the following:

1. The open interval $(-5, 5)$ is open in $(\mathbb{R}, d_{std})$.
2. The set $\mathbb{R}$ is open in $(\mathbb{R}, d_{std})$.
3. The set $U = \mathbb{R} \setminus \mathbb{Z}$ (of all numbers that are not integers) is open in $(\mathbb{R}, d_{std})$.
4. The set $U = \mathbb{R}^2 \setminus \mathbb{Z}^2$ (of all points in the plane whose coordinates are not both integers) is open in $(\mathbb{R}^2, d_{std})$.

1.10 Open subsets, II

Verify the following:

1. The set $\mathbb{Q}$ (of rational numbers) is not open in $(\mathbb{R}, d_{std})$.
2. The set $\mathbb{R} \setminus \mathbb{Q}$ (of irrational numbers) is not open in $(\mathbb{R}, d_{std})$.
3. The set $\mathbb{R}^2 \setminus S^1$ (of all points not on the unit circle) is open in $(\mathbb{R}^2, d_{std})$.