Extra Credit Assignment 2

Due Thursday, September 10, 11:59 PM

Preliminaries on complex numbers

A *complex* number is an expression of the form $x + iy$, where x and y are real numbers, and i is a square root of -1.[1] We let $\mathbb{C}$ denote the set of all complex numbers. There is a bijection

$$\mathbb{C} \to \mathbb{R}^2, \qquad x + iy \mapsto (x, y).$$

If you've ever learned about the "complex plane," or how to visualize a complex number as an element of the xy-plane, you have been using this bijection to visualize complex numbers.

We often write a complex number as z or as w.

Given a complex number $w = x + iy$, we let its *complex conjugate* be

$$\overline{w} = x - iy.$$

Finally, the product of two complex numbers is dictated by algebra. If $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$, then

$$w_1 w_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

You should work this out if you haven't before; make sure to use that $i^2 = -1$. You should also write out the product $w\overline{w}$ in terms of x and y. It should look familiar.

S^3 consists of certain pairs of complex numbers

Note that the bijection $\mathbb{C} \cong \mathbb{R}^2$ induces a bijection $\mathbb{C}^2 \cong \mathbb{R}^4$. (So you can think an element of $\mathbb{C}^2$—that is, a pair of complex numbers—as a four-dimensional vector.) You can also check that the set

$$\{(w_1, w_2) \,|\, w_1 \overline{w_1} + w_2 \overline{w}_2 = 1\} \subset \mathbb{C}^2$$

is sent to $S^3 \subset \mathbb{R}^4$ under this bijection. In what follows, we will use this implicitly, identifying points of S^3 with (certain) pairs of complex numbers.

[1] No real number can be a square root of -1; that is why we have to create a new number called i. Note that i is only one square root of -1. The other square root is $-i$.

The assignment:

Here is an interesting function:

$$p : S^3 \to \mathbb{C} \times \mathbb{R}, \qquad (w_1, w_2) \mapsto (2w_1\overline{w_2}, w_1\overline{w_1} - w_2\overline{w_2}).$$

(a) Using the fact that $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$, tell me why the image of p is S^2. If you convince me of this, p can then be considered a function from S^3 to S^2.

(b) Choose any point $a \in S^2$. Tell me why the pre-image $p^{-1}(a)$ is, or looks like, a circle. Hint: It may help to note that the set of complex numbers w such that $w\overline{w} = 1$ can be identified with a circle, and you can multiply pairs (w_1, w_2) by such w to obtain a new pair (ww_1, ww_2).

(c) Based on this, do you think that there is a bijection between S^3 and $S^1 \times S^2$? Can you exhibit such a bijection? Is it continuous? (It's okay if you don't know what "continuous" means; but can you exhibit a bijection that "feels" continuous?)