Proof Assignment 2

Due Tuesday, September 8, 11:59 PM

This week, you'll practice thinking about intersections and unions.

Notation

Let $\mathcal{S}$ be a collection of some sets. (In many of our examples, $\mathcal{S}$ will be the power set of some set.) Let $\mathcal{A}$ be a set. And fix a function $S : \mathcal{A} \to \mathcal{S}$. Out of laziness, if $\alpha \in \mathcal{A}$, instead of writing $S(\alpha)$, we will write S_α for the value of S on α. Various authors use different notation for this idea:

(a) Instead of writing out S as a function, they may write $\{S_\alpha\}_{\alpha \in \mathcal{A}}$, i.e., a collection of sets indexed by α.

(b) They may simply say, "choose a set S_α for each $\alpha \in \mathcal{A}$."

Using the "a set is a bag with stuff in it" language, the data of $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ is simply a chosen array of bags. How many bags are there? There are as many bags as there are elements in $\mathcal{A}$. Note that some bags may be repeated—S may be non-injective.

Warning 0.0.1. Keep in mind that $\mathcal{A}$ may be an infinitely large set!

Then

$$\bigcup_{\alpha \in \mathcal{A}} S_\alpha \tag{0.0.0.1}$$

is the *union* of all the sets S_α. In other words, we take all the bags and dump their contents into a single large bag. This single large bag is written $\bigcup_{\alpha \in \mathcal{A}} S_\alpha$, as in (0.0.0.1). Be warned that some bags may contain "the same elements," and these elements are *not* double-counted, or doubled-up, in the big bag. This is just the usual notion of union of sets.

Likewise,

$$\bigcap_{\alpha \in \mathcal{A}} S_\alpha$$

is the *intersection* of all the sets S_α. In other words, we look through all the bags, and determine if there are elements that are contained in every bag in sight. We form a new bag, $\bigcap_{\alpha \in \mathcal{A}} S_\alpha$, consisting only of those elements that appear in every bag.

Proof problem

Let $(P, \leq)$ be a poset. We will call a subset $U \in \mathcal{P}(P)$ an *open* subset of P if and only if the following holds:

$$\text{If } p \in U \text{ and } p \leq q, \text{ then } q \in U.$$

In other words, $U \subset P$ is open if and only if it is "upward closed."

Example 0.0.2. The empty set $\emptyset \subset P$ is open. The entire set P is open.

Here is the problem:

(a) Let $U : \mathcal{A} \to \mathcal{P}(P)$ be a function such that, for every $\alpha \in \mathcal{A}$, U_α is an open subset of P. Show that

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$

is open. (In words: You are showing that the union of open subsets is open.)

(b) Let $U : \mathcal{A} \to \mathcal{P}(P)$ be a function such that, for every $\alpha \in \mathcal{A}$, U_α is an open subset of P. Show that

$$\bigcap_{\alpha \in \mathcal{A}} U_\alpha$$

is open. (In words: You are showing that the intersection of open subsets is open.)

(c) Let P and Q be posets. A function $f : P \to Q$ will be called *continuous* if and only if the preimage of any open subset of Q is an open subset of P. In other words, if $V \subset Q$ is open, then $f^{-1}(V)$ is open in P. Prove that f is continuous if and only if it is a map of posets.

Canvas Questions:

This week, the Canvas questions will be multiple choice.

Below, you are given a set $\mathcal{A}$, and for each $\alpha \in \mathcal{A}$, you are given a set U_α. For each of these, you should be able to identify the sets $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ and $\bigcap_{\alpha \in \mathcal{A}} U_\alpha$ with one of the sets (A) - (G).

(I) Let $\mathcal{A} = \{\alpha > 0\} \subset \mathbb{R}$, and for each $\alpha \in \mathcal{A}$, let

$$U_\alpha = (-\alpha, \alpha) \times (-\alpha, \alpha) \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2.$$

(II) Fix a positive real number δ. Let

$$\mathcal{A} = \{(x, r) | r > 0 \text{ and } \sqrt{x_1^2 + x_2^2} + r < \delta\} \subset \mathbb{R}^2 \times \mathbb{R}$$

and for each $\alpha = (x, r) \in \mathcal{A}$, let

$$U_\alpha = \{y \in \mathbb{R}^2 \,|\, \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < r\} \subset \mathbb{R}^2.$$

(III) Fix a positive real number δ. Let

$$\mathcal{A} = \{(x, r) \,|\; |x_1| < \delta, |x_2| < \delta,\ r > 0, \text{ and } |x_1| + |x_2| + r < \delta\}.$$

We let

$$U_\alpha = \text{Ball}(x, r).$$

Here are some sets:

(A) $\emptyset$

(B) $\text{Ball}(0, \delta) \subset \mathbb{R}^2$

(C) The closed ball, in $\mathbb{R}^2$, of radius δ centered at 0.

(D) $(-\delta, \delta) \times (-\delta, \delta) \subset \mathbb{R}^2$.

(E) $[-\delta, \delta] \times [-\delta, \delta] \subset \mathbb{R}^2$.

(F) The set $\{(0, 0)\} \subset \mathbb{R}^2$ consisting only of the origin.

(G) $\mathbb{R}^2$.