Homework 3

Due Tuesday, September 15, 11:59 PM

Proof problem

Preliminaries Let $\mathbb{Z}_{>0} \subset \mathbb{Z}$ be the set of all *positive* integers. (So elements are 1, 2, 3, et cetera.)

Let A be a set. A *sequence* in A is a function from $\mathbb{Z}_{>0}$ to A. Often, we will call this function a, and let a_i denote the image of $i \in \mathbb{Z}_{>0}$. It is common to denote a sequence by the notation

$$(a_i)_{i=1}^{\infty}, \qquad \text{or} \qquad \{a_i\}_{i=1}^{\infty} \qquad \text{or} \qquad a_1, a_2, a_3, \ldots.$$

(I mention all these different notations just in case you've seen one of them before.)

Given $x, y \in \mathbb{R}^n$, we will let the distance from x to y be denoted $d(x, y)$. This is given by the formula

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}.$$

Finally, let $A = \mathbb{R}^n$. We say that a sequence $a_1, a_2, \ldots$ in $\mathbb{R}^n$ *converges* to a point $b \in \mathbb{R}^n$ if and only if the following holds:

For every positive real number ϵ, there exists a positive integer N so that

$$i > N \implies d(a_i, b) < \epsilon.$$

The problem Fix a subset $K \subset \mathbb{R}^n$. Prove that the following two statements are equivalent:

(a) The complement[2] of K is an open subset of $\mathbb{R}^n$.

(b) For any sequence $a_1, a_2, \ldots$ in[3] K such that the sequence converges to some $b \in \mathbb{R}^n$, b is in fact an element of K.

[2] That is, the set $\mathbb{R}^n \setminus K$

[3] So every a_i is an element of K

In other words, prove that K satisfies (a) if and only if K satisfies (b). These give two equivalent ways to think about the *closedness* of a set K.

Hint: For this problem, you will need the triangle inequality, which says that for three points $x, y, z \in \mathbb{R}^n$, we have

$$d(x, y) + d(y, z) \geq d(x, z).$$

You will *not* need to invoke the *formula* for distance. (Though, of course, in life, we have had to use that formula to prove the triangle inequality.) Put another way, you will not need to prove the triangle inequality, but you may use it.

Canvas Questions:

You should be able to decide whether each of the following subsets of $\mathbb{R}^n$ is closed, open, both, or neither:

(I) The subset $\mathbb{R}^n$ itself.

(II) The subset $\emptyset \subset \mathbb{R}^n$.

(III) In $\mathbb{R}^2$, the set consisting of only the origin.

(IV) In $\mathbb{R}^1$, the open interval from -1 to 1.

(V) In $\mathbb{R}^2$, the set $\{(x_1, x_2) \,|\, x_1 = 0 \text{ and } -1 < x_2 < 1\}$.

(VI) In $\mathbb{R}^2$, the set $\{(x_1, x_2) \,|\, x_1 = 0 \text{ and } -1 \leq x_2 \leq 1\}$.