Homework 4

Due Tuesday, September 22, 11:59 PM

Proof problem

For this problem, you will need to know what a *topology* on a set X is. (This is in the notes.) I will write a topology as $\mathcal{T}$. Remember that $\mathcal{T}$ is a subset of $\mathcal{P}(X)$ satisfying some properties, and a given set X may admit many different topologies!

Fix a set X.

(a) Show that $\mathcal{P}(X)$ itself is a topology on X. (That is, show that $\mathcal{T} = \mathcal{P}(X)$ is a topology.)

(b) For any collection $\{\mathcal{T}_\beta\}_{\beta \in \mathcal{B}}$ of topologies, show that the intersection

$$\bigcap_{\beta \in \mathcal{B}} \mathcal{T}_\beta$$

is a topology on X.

(c) Let $\mathcal{S} \subset \mathcal{P}(X)$ be a collection of subsets of X. ($\mathcal{S}$ is not necessarily a topology.) Show that $\mathcal{S}$ is contained in some topology on X. (Hint: Given the first part of this problem, this problem is easy once you understand what it's asking.)

(d) Again let $\mathcal{S} \subset \mathcal{P}(X)$ be a collection of subsets of X. Let

$$\mathcal{B} := \{\mathcal{T}' \subset \mathcal{P}(X) \,|\, \mathcal{T}' \text{ is a topology on } X, \text{ and } \mathcal{S} \subset \mathcal{T}'.\}$$

and

$$\mathcal{T}_\mathcal{S} := \bigcap_{\mathcal{T}' \in \mathcal{B}} \mathcal{T}'.$$

Show that $\mathcal{S} \subset \mathcal{T}_\mathcal{S}$ and that for any topology $\mathcal{T}'$ containing $\mathcal{S}$, that $\mathcal{T}_\mathcal{S} \subset \mathcal{T}'$.

Canvas Quiz Questions:

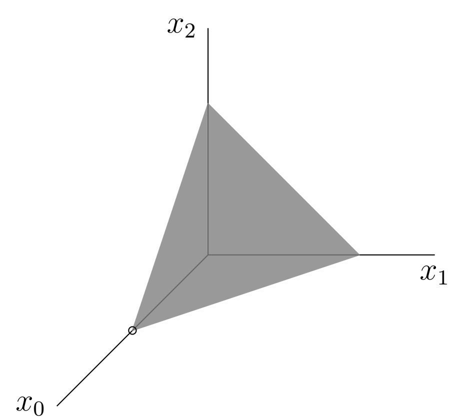
Let $X = \Delta^n$, and $Y = [n]$. Define

$$f : X \to Y, \qquad x \mapsto \text{The largest } i \text{ such that } x_i > 0.$$

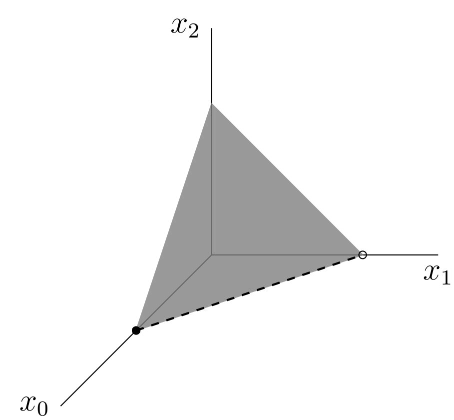
That is, writing $x = (x_0, x_1, \ldots, x_n)$[4], f sends x to the largest number i for which the coordinate x_i is a positive number.

For this problem, we will specialize to $n = 2$. For each V below, you should be able to tell me which of the images below describes the subset $f^{-1}(V)$ inside Δ^2.

1. $V = \{0\}$.
2. $V = \{1\}$.
3. $V = \{2\}$.
4. $V = \{1, 2\}$.
5. $V = \{0, 1\}$.
6. $V = \{0, 1, 2\}$.

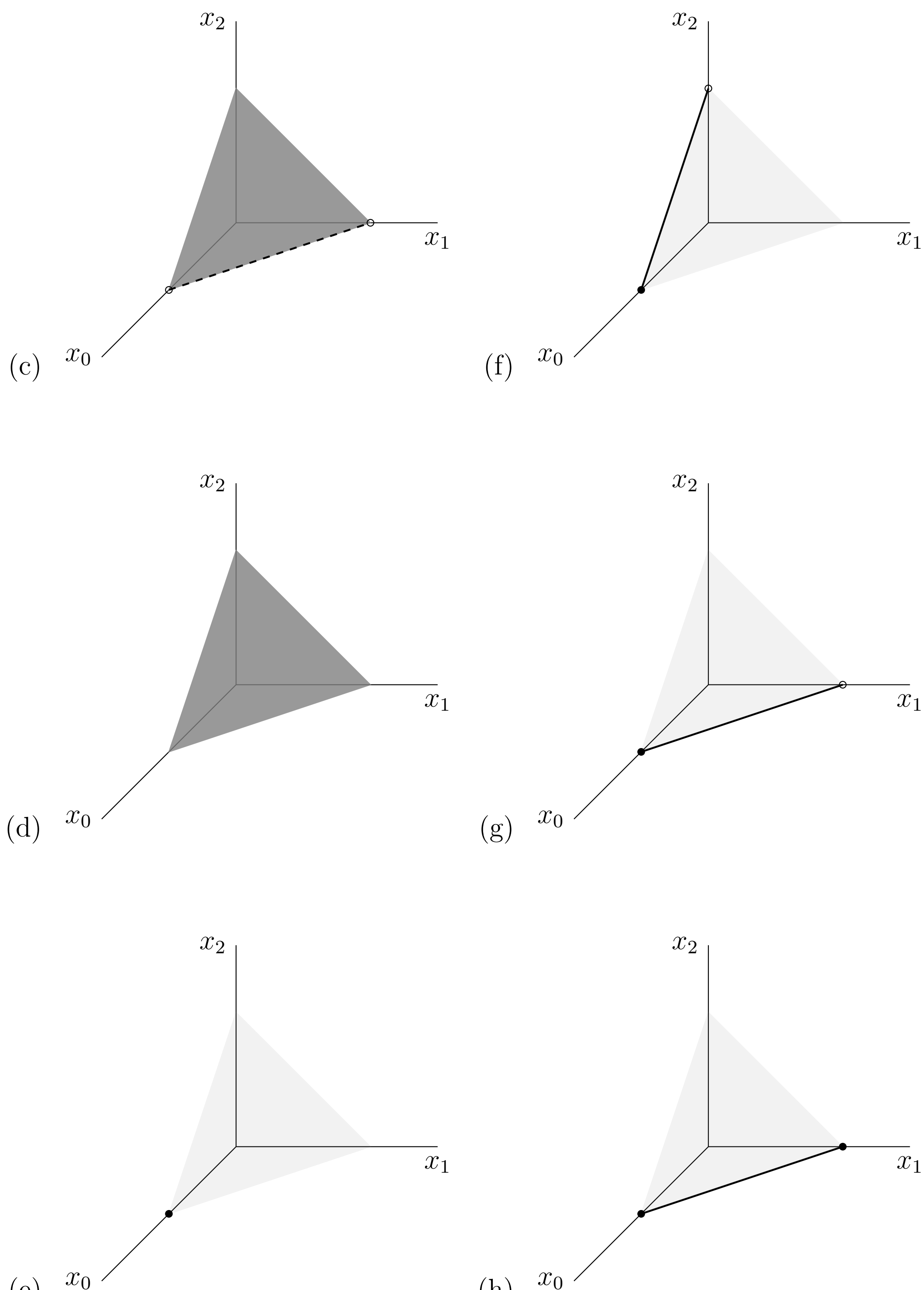


(a)

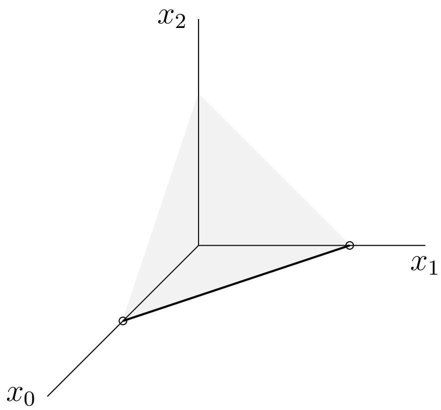


(b)

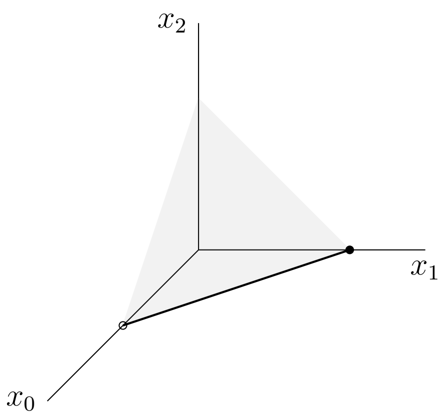
[4]Recall that when defining $\Delta^n \subset \mathbb{R}^{n+1}$, we index the coordinates from 0 to n.



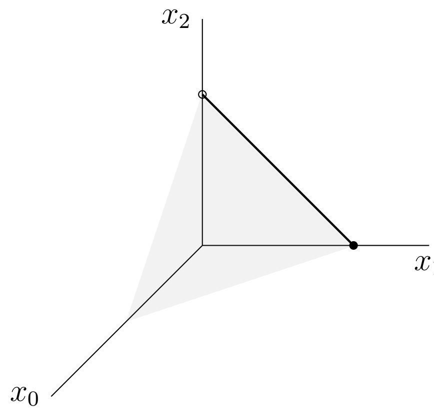
(c)
(d)
(e)
(f)
(g)
(h)
x_0
x_1
x_2



(i)



(j)



(k)