Lecture 1

Sets and introductions

Let's first review some common conventions about how we talk about *sets* as mathematicians. The review will not be extensive, as you are assumed (at this university) to have already taken a class on basics of sets and proof.

1.1 Sets and elements

Recall that a *set* is a collection of objects. If A is a set, an *element of* A is an object in A.

Notation 1.1.1. If (and only if) a is an element of A, we write $a \in A$. If multiple elements a, a', and a'' are in A, we may write $a, a', a'' \in A$.

We may sometimes say that "a is in A" or "a is contained in A" as well.

Example 1.1.2. Let $\mathbb{Z}$ be the set of all integers. Then the integer 3 is an element of $\mathbb{Z}$. We write $3 \in \mathbb{Z}$.

Let $\mathbb{Q}$ be the set of all rational numbers. (Numbers that can be expressed as fractions of integers.) Then $3 \in \mathbb{Q}$, and $2/5 \in \mathbb{Q}$.

Let $\mathbb{R}$ be the set of all real numbers. Then $3, 2/5, \pi, \sqrt{2}, e \in \mathbb{R}$.

Remark 1.1.3. It is very common to denote sets by capital letters, and to denote elements of that set by the same letter, but lower-case. For example, you will often read: "Let A be a set, and fix $a \in A$."

While this convention is not always followed, I would encourage it. The major exception to this convention is when sets are "famous," or already come with a set notation, or have a different font. For example, $\mathbb{R}$ is the set of real numbers, but we more commonly write x, y, t for real numbers than r. Likewise, we rarely write z for an integer; instead using letters like a, b, m, n, i, j.

Warning 1.1.4. We have not defined (in this course) what a *set* actually is. This is very tricky business, and would take a long time to set up sets accurately and precisely. See Exercises 1.7.1 and 1.7.2. In this course, we bring your attention to this danger as a boogeyman, then hereafter ignore it.

Notation 1.1.5. We will write

$$A = \{5, 7, 8\}$$

to mean that A is a set consisting of exactly three elements, called 5, 7, and 8. Likewise, the notation

$$B = \{\text{Bob, banana, Alice, } 1\}$$

means that B is a set consisting of exactly four elements, called Alice, Bob, 1, and banana.

1.2 Subsets

Let B be a set. We say that A is a *subset of* B if (and only if) every element of A is an element of B.

If A is a subset of B, we will write

$$A \subset B.$$

Example 1.2.1. Let $A = \{1, 3, 5\}$. Then A is a subset of $\mathbb{Z}$.

Intuition 1.2.2. Let B be a set. You might imagine that B is a bag containing things, and an *element* of B is simply a thing in the bag.

Now imagine dumping the contents of B into a water tank. Take a big ladle, or some sort of scoop, and take a big scoop. Whatever you collect in your ladle is an example of a subset of B.

1.3 The empty set

The *empty set* is the set containing no elements. We denote the empty set by the symbol

$$\emptyset.$$

Intuition 1.3.1. Think back to Intuition 1.2.2. Note that, when you scoop, you might come out empty-handed (you might scoop up nothing)! This is an example of the empty set.

This thought experiment/intuition is supposed to reinforce the idea that *the empty set is a subset of any set*. Put another way, for any set B, we may write $\emptyset \subset B$.

1.4 Proving two sets are equal

We say that two sets A and B are *equal* if they have exactly the same elements, and we will write

$$A = B$$

when two sets are equal.

The most common way to prove that two sets A and B are equal is to show that $A \subset B$ *and* $B \subset A$.

Conversely, if $A = B$, then $A \subset B$ and $B \subset A$.

1.5 Sets and whole numbers

A lot of students get confused about the relationship between sets and numbers.

Given any set, we can ask "how many elements are in the set?" We can ask it, but we may not always get an answer we're familiar with. This is because "how many" is a question that is usually answered by a whole number, like 0, 1, 2, 3, et cetera. If you have taken a course where you learned about the term *cardinality*, you know that there are even bigger kinds of numbers, like different kinds of infinity, which cannot be expressed by a whole number.

For example, "how many elements are in the set $\mathbb{Z}$" and "how many elements are in the set $\mathbb{R}$" are questions that turn out to have *different* answers, because while both sets are infinitely large, one can prove that the cardinality of $\mathbb{R}$ is strictly larger than the cardinality of $\mathbb{Z}$.

Of course, two different sets may have the exact same number of elements, but the sets may be unequal. For example, if A is a set containing three apples, and B is a set containing three bananas, then A and B have the same cardinality, but they are clearly different sets.

1.6 In-class exercises: The power set

Let A be a set. The *power set* of A is the set

$$\mathcal{P}(A) := \{B \subset A\}.$$

That is, $\mathcal{P}(A)$ is the set of all subsets of A.

Exercise 1.6.1.

1. Let A_0 be the empty set. How many elements are in A_0?
2. Write out all the elements of $\mathcal{P}(A_0)$.
3. How many elements are in $\mathcal{P}(A_0)$?

Exercise 1.6.2.

1. Let A_1 be the set containing a single element called a.
2. Write out all the elements of $\mathcal{P}(A_1)$.
3. How many elements are in $\mathcal{P}(A_1)$?

Exercise 1.6.3.

1. Let A_2 be the set containing exactly two elements, called a and b.
2. Write out all the elements of $\mathcal{P}(A_2)$.
3. How many elements are in $\mathcal{P}(A_2)$?
4. What does $\mathcal{P}(A_2)$ have to do with squares?

Exercise 1.6.4. Let A_3 be the set containing exactly three elements, which we call a, b, and c. What does $\mathcal{P}(A_3)$ have to do with cubes?

Exercise 1.6.5. Fix an integer $n \geq 0$. Let A be a set containing exactly n elements. How many elements do you think are in $\mathcal{P}(A)$? Can you prove it?

If a square is a "two-dimensional" analogue of a cube, and a cube is a "three-dimensional" analogue of a cube, do you have any ideas as to how you might draw the four-dimensional analogue of a cube?

1.7 Exploratory exercises (optional)

Exercise 1.7.1. In this exercise, you will study the notion of "a set of all sets." Your study will give strong evidence (to you) that we must be careful what we mean by a set. In this course, we will introduce this danger as a boogeyman, then hereafter ignore it.

Exercise: Study the following proof. What, if anything, is wrong with it?

Let X be the set of all sets, and let $\mathcal{P}(X)$ be the power set. By definition of X, X must contain $\mathcal{P}(X)$. But by Cantor's Theorem, the power set of any set has strictly higher cardinality than the original set—so X could not contain $\mathcal{P}(X)$.

We have thus begun with the assumption that the set of all sets exists, and ended with a conclusion.

Exercise 1.7.2. Here is another exercise showing how careful we must be when defining sets.

Exercise: Study the following proof. What, if anything, is wrong with it?

Let X be the set of all sets that do not contain themselves as an element. For example, the set $A = \{2, 3\}$ does not contain itself.[1] We may ask whether X is in itself. If $X \in X$, we arrive at a contradiction due to the definition of X. If $X \notin X$, then by definition of X, we again arrive at a contradiction.

[1] You may now wonder—is there a set that contains itself? This question gets to the heart of the "foundations" of mathematics. The most commonly used axioms of set theory exclude such a possibility, in that one can prove from the axioms that no set can contain itself. However, there are other axioms of set theory that do allow for sets that contain themselves.